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Do We Have a Sense for Irrational Numbers?

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Abstract

Number sense requires, at least, an ability to assess magnitude information represented by number symbols. Most educated adults are able to assess magnitude information of rational numbers fairly quickly, including whole numbers and fractions. It is to date unclear whether educated adults without training are able to assess magnitudes of irrational numbers, such as the cube root of 41. In a computerized experiment, we asked mathematically skilled adults to repeatedly choose the larger of two irrational numbers as quickly as possible. Participants were highly accurate on problems in which reasoning about the exact or approximate value of the irrational numbers' whole number components (e.g., 3 and 41 in the cube root of 41) yielded the correct response. However, they performed at random chance level when these strategies were invalid and the problem required reasoning about the irrational number magnitudes as a whole. Response times suggested that participants hardly even tried to assess magnitudes of the irrational numbers as a whole, and if they did, were largely unsuccessful. We conclude that even mathematically skilled adults struggle with quickly assessing magnitudes of irrational numbers in their symbolic notation. Without practice, number sense seems to be restricted to rational numbers.

Keywords: number sense, magnitude representation, number comparison, natural number bias, numerical distance effect

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Number sense refers to the adaptive and flexible use of numbers. A precondition of number sense is the ability to quickly assess magnitude information represented by number symbols (e.g., 12, -5, or 3/4). Curricula and educational standards for school mathematics (e.g., CCSSI, 2010) include an understanding of number magnitudes as an important goal of school mathematical learning. Furthermore, theories from cognitive psychology consider understanding of number magnitudes as a fundamental step of numerical development (Siegler, Thompson, & Schneider, 2011; see Siegler, Fazio, Bailey, & Zhou, 2013). The reason why understanding number magnitudes is considered so important is that possessing magnitudes is a shared property of all types of numbers which students learn about at school, whether natural, whole, rational, or real numbers.[i] Thus, magnitudes are a key feature of all real numbers.

In spite of the widely accepted importance of number magnitude understanding, research into magnitude understanding has almost exclusively focused on natural numbers (i.e., the counting numbers) (e.g., De Smedt, Verschaffel, & Ghesquière, 2009), integers (including negative numbers, e.g., Varma & Schwartz, 2011; Young & Booth, 2015), and rational numbers in decimal notation (e.g., 0.2) (Desmet, Grégoire, & Mussolin, 2010) or fraction notation (e.g., 1/5) (Van Hoof, Lijnen, Verschaffel, & Van Dooren, 2013). Studies have amply shown that most people are able to assess magnitudes of these numbers, albeit with more difficulty and less automatically for larger and less common numbers (e.g., fractions with two-digit components such as 31/73) than for smaller numbers and numbers that are used frequently in daily life (e.g., 1/4) (Bonato, Fabbri, Umiltà, & Zorzi, 2007; Obersteiner, Van Dooren, Van Hoof, & Verschaffel, 2013).

To the best of our knowledge, no study has investigated whether people are also able to assess magnitudes of irrational numbers in their symbolic notation format, such as $\sqrt[3]{41}$, the cube root of 41. Addressing this question is relevant to cognitive psychology, because it contributes to the more general question of whether our cognitive system allows assessing magnitudes of all types of numbers (including non-rational numbers). Addressing this question has also the potential to inform mathematics education, because although the ability to quickly assess the magnitudes of number symbols is a central goal of instruction, the boundaries of this goal are not obvious. For example, while teachers might expect students to be able to provide a quick estimate for the magnitudes of the square root of 2 or the number π, it is not clear whether they should expect them to be able to provide an estimate for the square root of 3, or of the fifth root of 82.

Accordingly, this study aims to clarify whether without specific practice, people are able to assess the magnitudes of irrational numbers presented in symbolic format. In the following sections, we first elaborate on the importance of understanding number magnitudes. Then, we explain the peculiarities of irrational numbers and their symbolic notation. After that, we introduce the number comparison task as a measure of magnitude understanding and summarize important findings from studies of whole numbers and fractions that were guiding for the present study. Finally, we raise open questions that we address here.

The Importance of Understanding Number Magnitudes

Possessing "number sense" refers to the ability to use numbers flexibly and adaptively (e.g., Dehaene, 1997; McIntosh, Reys, & Reys, 1992). Different authors have used the term "numbers sense" in various ways (see Berch, 2005, for an overview). In its narrow conceptualization, number sense refers to the ability to understand the meaning of numbers in their nonsymbolic representations (e.g., dot patterns) and symbolic representations. In its broader conceptualization, number sense includes the use of sophisticated strategies in complex arithmetic. Irrespective of the specific conceptualization, at the core of *symbolic* number sense is the ability to quickly assess the magnitudes represented by number symbols. This ability is a precondition for the flexible and adaptive use of numbers for at least three reasons. First, quickly activating number magnitudes can help in choosing the most efficient strategy for solving an arithmetic problem. For example, to find the result of the subtraction problem 701 – 698, some reasoning about the magnitudes of the two numbers (both close to 700) is required to decide that adding up from 698 to 701 (resulting in 3) is more efficient than actually subtracting 698 from 701. Second, quick assessment of number magnitudes allows us to reject unreasonable results of a calculation. For instance, when students apply an invalid addition strategy to fractions, such as componential addition (e.g., 1/2 + 1/2 = 2/4), quick reasoning about the fraction magnitudes can cause them to doubt their result (because the result, 2/4, has the same magnitude as each addend, 1/2). Third, assessing number magnitudes is important for estimation and approximation. To quickly come up with an estimate for the addition problem 7/8 + 12/13 (which is approximately 2), it is necessary to recognize that the magnitudes of both fractions are close to 1.

There is broad empirical evidence that the ability to assess magnitudes of whole numbers and fractions is correlated with and even predictive of further mathematical achievement (De Smedt et al., 2009; Linsen, Verschaffel, Reynvoet, & De Smedt, 2014, 2015; Sasanguie, De Smedt, Defever, & Reynvoet, 2012; Siegler et al., 2012). Torbeyns, Schneider, Xin, and Siegler (2015) found that assessing magnitudes of fractions is significantly correlated to mathematics achievement in 6^{th}- and 8^{th}-grade students from the U.S., China, and Belgium, even after controlling for several other related variables. In a recent meta-analysis, Schneider et al. (2016) concluded that symbolic magnitude understanding correlates to mathematical competence more strongly than nonsymbolic magnitude understanding.

To summarize, the ability to assess magnitudes of number symbols is fundamental for numerical development from both a theoretical and an empirical point of view. While previous empirical studies have focused exclusively on magnitudes of *rational* numbers (including natural numbers, integers, and fractions), some researchers have emphasized the importance of understanding the magnitudes of *any real numbers* (which include irrational numbers) (Siegler et al., 2011; Torbeyns et al., 2015). For example, the integrated theory of numerical development by Siegler et al. (2011, p. 289) puts emphasis "on acquisition of knowledge about numerical magnitudes as a basic process uniting the development of understanding of all real numbers". Referring to the same theory, Torbeyns et al. (2015, p. 12) conclude that "a key step in understanding real numbers is the realization that not only whole numbers or rational numbers but in fact all real numbers have magnitudes that can be represented along a number line". While these authors refer to *understanding* of real number magnitudes, there is—to the best of our knowledge—no empirical evidence that people are actually able to quickly *assess* magnitudes of these numbers when they are presented in their symbolic format. As irrational numbers differ in important ways from rational numbers, assessing their magnitudes might be impossible, or at least much more challenging than is the case for rational numbers.

Magnitudes of Irrational Numbers

Irrational numbers are real numbers that cannot be represented as the quotient of two integers. These numbers differ from rational numbers in several respects, two of which are relevant in the context of the present study. First, while for rational numbers the magnitude information can be decoded from their symbolic notation in a more or less straightforward way, this is not the case for irrational numbers. Assessing the magnitudes of integers requires an understanding of the magnitude information of numerals, an ability to take into account the base-ten system, and, if applicable, an understanding of the minus sign. Assessing the magnitudes of fractions requires an understanding of the magnitude information of the whole number components (the numerator and the denominator) and some reasoning about the relation between these magnitudes. In contrast, the algorithm to determine the magnitude of an irrational number (e.g., $\sqrt[3]{41}$) is complex. In fact, such an algorithm consists of an infinite number of steps. The reason is related to the second aspect in which irrational numbers differ from most types of rational numbers: it is not possible to notate the exact value of irrational numbers in decimal notation because they have an infinite number of non-recurrent digits after the decimal point.[ii] For example, the square root of 2 is approximately 1.4142135…. Because of these differences, and because people arguably rarely encounter irrational numbers in their daily lives, it should be much more difficult to assess the magnitudes of irrational numbers than those of integers or fractions. For that reason, we studied people with good mathematical skills, to find out whether it is *at all* possible to assess magnitudes of irrational numbers.

Number Comparison Task as a Measure of Magnitude Understanding

To investigate whether people are able to assess magnitudes of number symbols, many studies have used a number comparison task. In this task, participants have to decide which of two numbers is numerically larger.

Next to participants' accuracy when solving number comparison problems, the occurrence of a numerical distance effect across a set of problems is of particular interest. The distance effect means that people tend to be more accurate and faster in comparing the numerical values of two numbers when the numerical difference between the two numbers is large compared to when it is small. Moyer and Landauer (1967) initially documented this effect, and many researchers replicated this effect with whole numbers in a variety of studies involving participants of different ages (De Smedt et al., 2009; Sekuler & Mierkiewicz, 1977; Szűcs & Goswami, 2007). The occurrence of a distance effect provides evidence that people actually assess the magnitudes of the number symbols to solve number comparison problems.

In fraction comparison problems, the central question is whether accuracy and response times depend on the numerical distance between the fraction magnitudes, or between the fraction components (the numerators and the denominators). While the former is evidence that people assess the fraction magnitudes, the latter is evidence that people assess the magnitudes of the components. Recent research has converged in the finding that whether people assess the magnitudes of fractions in a comparison problem depends on the type of fraction pair. When the two fractions have common components (e.g., 3/7 versus 5/7, or 5/9 versus 5/6), people are more likely to rely on the non-equal fraction components rather than on the magnitudes of the whole fractions. When fractions do not have common components (e.g., 5/9 versus 6/11), people are more likely to assess fractions according to their magnitudes rather than their components alone (Huber, Moeller, & Nuerk, 2014; Meert, Grégoire, & Noël, 2009, 2010a; Obersteiner et al., 2013; Schneider & Siegler, 2010).

There is only a limited amount of research on people's understanding of irrational numbers (Fischbein, Jehiam, & Cohen, 1995; Merenluoto & Lehtinen, 2002; Peled & Hershkovitz, 1999; Sirotic & Zazkis, 2007; Zazkis, 2005). A central finding from these studies is that many people, even preservice teachers and teachers of mathematics, struggle with understanding certain concepts of real numbers including irrational numbers. However, these studies focused on understanding of the concept of irrational numbers in a broader sense, but none of these studies has particularly focused on the ability to assess magnitudes of irrational numbers.

Natural Number Bias in Number Comparison

Notwithstanding most people's fundamental ability to assess fraction magnitudes in comparison problems, assessing magnitudes of fractions seems to be much more demanding and less automatic than assessing magnitudes of whole numbers. For example, comparing fractions that have common components (so that assessing the magnitudes of the components is sufficient) is much easier than comparing fraction pairs without common components (where assessing the magnitudes of the components is not sufficient) (Obersteiner et al., 2013). Furthermore, there is evidence that in fraction comparison, the magnitudes of the fractions' whole number components can actually interfere with assessing the fraction magnitudes. Research shows that school students (Meert, Grégoire, & Noël, 2010b; Van Hoof et al., 2013) but also adults (DeWolf & Vosniadou, 2011; Obersteiner et al., 2013; Vamvakoussi, Van Dooren, & Verschaffel, 2012) make more mistakes and require more time for fraction comparison when the magnitudes of the fractions are incongruent with whole number comparison as compared to when they are congruent with whole number comparison. Comparison problems are incongruent when the larger fraction is composed of the smaller components, as in "19/24 vs. 25/36". Comparison problems are congruent when the larger fraction is composed of the larger components, as in "20/27 vs. 11/19". Researchers have referred to the performance differences between congruent and incongruent comparison problems as the "whole number bias" or "natural number bias" (Alibali & Sidney, 2015; DeWolf & Vosniadou, 2015; Ni & Zhou, 2005; Obersteiner et al., 2013). The existence of bias corroborates earlier findings suggesting that people activate natural number magnitudes automatically and unintentionally even if doing so is not necessary for solving the problem at hand (Hubbard, Piazza, Pinel, & Dehaene, 2005).

Research on fractions provides an important basis for studying the ability to assess magnitudes of irrational numbers that are represented as roots (e.g., $\sqrt[3]{41}$) because—just like fractions—these representations include two whole number components (e.g., 3 and 41). Accordingly, when comparing the magnitudes of two irrational numbers in that notation format, we can analyze the same behavioral patterns as we can with fractions. Specifically, we can analyze whether accuracy and speed depend on the irrational numbers' whole number components or on the magnitudes of the irrational numbers as a whole. Moreover, like with fractions, one of the whole number components (the radicand) of an irrational number is positively related to the overall magnitude (i.e., increasing the radicand makes the number greater), while the other whole number component (the index) is negatively related to the overall magnitude (i.e., increasing the index makes the number smaller). This way, irrational number comparison problems, just like fraction comparison problems, can be classified as being congruent (larger number is composed of larger components) or incongruent (larger number is composed of smaller components) with natural number comparisons. Accordingly, we can assess whether people show a natural number bias when comparing the magnitudes of irrational numbers. As in the case of fractions, an empirical indication of a natural number bias in irrational number comparison would be greater accuracies and lower response times on congruent compared to incongruent comparison problems.

The Present Study

We addressed the question of whether mathematically highly skilled adults can assess numerical magnitudes of irrational numbers, presented in their (exact) symbolic format[iii], to solve number comparison problems. We also aimed to characterize their reasoning process in such problems. We presented numbers in "root notation" (i.e., $\sqrt[n]{a}$), where the radicand *a* and the index *n* were natural numbers (i.e., we focused on algebraic numbers). Paralleling previous research on fraction comparison, we included comparison problems in which the irrational numbers either had common indices (e.g., $\sqrt[5]{93}$ vs. $\sqrt[5]{37}$), common radicands (e.g., $\sqrt[5]{43}$ vs. $\sqrt[8]{43}$), or no common whole number components (e.g., $\sqrt[7]{13}$ vs. $\sqrt[8]{21}$). Within both the common component (CC) and the no common component (No-CC) category, problems could be congruent (CO) or incongruent (IC) in terms of whole number comparison. In congruent problems, the larger number is composed of the larger component(s), while in incongruent problems, the larger number is composed of the smaller component(s). Problems without common components, in which the larger number is composed of the larger radicand and the smaller index, are in this sense neutral (N) because comparing the indices and comparing the radicands lead to contradictory results. Table 1 provides an overview of the types of comparison problems we used in our study.

One can solve comparison problems with common components easily by comparing the components while not assessing the number magnitudes. This is also true for the neutral problems with no common components, because the number with the larger radicand and the smaller index is always the larger number. In contrast, congruent and incongruent problems without common components require assessing the magnitudes of the irrational numbers, because componential comparison does not allow for a valid conclusion.[iv]

Table 1

Overview of the Different Problem Types

	Common Components	No Common Components
Congruent	**CC-CO** Index1 = Index 2 Radicand 1 > Radicand 2 Number 1 > Number 2 Example: $\sqrt[3]{21}$ vs. $\sqrt[3]{14}$	**No-CC-CO** Index 1 > Index 2 Radicand 1 > Radicand 2 Number 1 > Number 2 Example: $\sqrt[8]{84}$ vs. $\sqrt[5]{12}$
Incongruent	**CC-IC** Radicand 1 = Radicand 2 Index 1 > Index 2 Number 1 < Number 2 Example: $\sqrt[7]{48}$ vs. $\sqrt[5]{48}$	**No-CC-IC** Index 1 > Index 2 Radicand 1 > Radicand 2 Number 1 < Number 2 Example: $\sqrt[9]{74}$ vs. $\sqrt[6]{67}$
Neutral	–	**No-CC-N** Index 1 < Index 2 Radicand 1 > Radicand 2 Number 1 > Number 2 Example: $\sqrt[7]{78}$ vs. $\sqrt[8]{52}$

Note. CC = common components, No-CC = no common components, CO = congruent, IC = incongruent, N = neutral.

Previous studies have shown that educated adults were fairly accurate about comparison problems with whole numbers and comparison problems with fractions. For example, the academic mathematicians in the study by Obersteiner et al. (2013) solved 97% of the fraction comparison problems correctly. As irrational number magnitudes are presumably much more difficult to determine, we expected that participants would have to rely on component comparison strategies whenever possible. As component strategies are less cognitively demanding, we expected accuracy to be very high when component strategies were applicable (i.e., for problems of types CC-CO, CC-IC, and No-CC-N), but substantially and significantly lower when component strategies were not applicable (i.e., for problems of types No-CC-CO and No-CC-IC). For the same reason, we expected response times to be shorter for CC-CO, CC-IC, and No-CC-N problems as compared to No-CC-Co and No-CC-IC problems (Hypothesis 1).

Furthermore, we expected to find a natural number bias, because participants would heavily rely on component strategies. That is, we expected to find significantly higher accuracy rates and lower response times for congruent as compared to incongruent problems with common components (Hypothesis 2a). We also expected to find higher accuracy rates and lower response times for congruent rather than incongruent problems without common components (Hypothesis 2b).

As in previous studies on fractions, we analyzed the distance effect as an indicator of comparison strategies. We distinguished between a distance effect for the magnitudes of the irrational numbers (hereafter: holistic distance effect) and a distance effect for the whole number components (i.e., the indices and the radicands; hereafter: component distance effect). We expected to find a holistic distance effect for problems of types No-CC-CO and No-CC-IC because component strategies are not successful in these cases (Hypothesis 3a). However, we did not expect to find a holistic distance effect for problems of the other types (Hypothesis 3b) because one can easily compare their magnitudes using component strategies. Theoretically, we would expect to find a component distance effect for problems of types CC-CO, CC-IC, and No-CC-N because we expected participants to rely on component comparison strategies rather than holistic strategies in these cases. However, since the distance effect for whole numbers is known to decrease with age and level of expertise (Sekuler & Mierkiewicz, 1977), it might be too small to be detectable in our sample of mathematically skilled adults. In fact, Obersteiner et al. (2013) did not detect a component distance effect in academic mathematicians for fraction comparison problems of which accuracy rates and response times strongly suggested that the participants were actually relying on the fraction components rather than on the fraction magnitudes. It was therefore an open question as to whether or not we would find a component distance effect for any type of comparison problem.

Methods

Participants

The participants in this study were 45 mathematically skilled adults (mean age: 34.6 years; 17 female, 28 male). They were recruited at the Mathematical Department of a university in Germany. Twenty of the participants had a Bachelor's degree in mathematics (15) or physics (5) and were graduate students majoring in mathematics or physics, respectively. Another 20 participants had a Master's degree in mathematics and were research assistants at the Mathematical Department of the university. Another five participants were professors of mathematics. Although the participants in our study certainly do not engage in assessing magnitudes of irrational numbers on a regular basis, we were confident that they had a sound concept of irrational numbers and were perfectly able to understand the meaning of the symbolic notation of irrational numbers.

Stimuli

We constructed 70 number comparison problems with irrational numbers that were represented as the *n*th root of *a* ($\sqrt[n]{a}$), the index *n* and the radicand *a* being one-digit or two-digit natural numbers. As in previous research on fraction processing, we created problems of each of the five different types detailed above (see Table 1; for a list of all problems we used in this study, see Appendix). There were 14 problems of each type. In the first two types of problems, the number pairs had common components, so that they were congruent (CC-CO) or incongruent (CC-IC) with respect to whole number comparison. For all other problems, the number pairs did not have common components. These problems without common components were either congruent (No-CC-CO), incongruent (No-CC-IC), or neutral (No-CC-N) with respect to whole number comparison. The mean numerical difference between the numbers of each pair was equal for all categories. This difference was always 0.20, which is the same as in the fraction comparison study by Obersteiner et al. (2013). Furthermore, we constructed the problems such that there were no correlations between the numerical distances among the two components and between each component and the holistic magnitudes (indices-radicands distances: $r(68) = -.01$, $p = .908$; indices-holistic distances: $r(68) = .00$, $p = .992$; radicands-holistic distances: $r(68) = .09$, $p = .449$).

Procedure

The participants worked on the problems individually in a quiet room at the university. The problems were presented on a Laptop, using E-Prime software (Schneider, Eschman, & Zuccolotto, 2002). The participants were instructed that they would see two numbers at a time, and that they should choose the greater one as quickly and accurately as possible by pressing the corresponding left (‘f’) or right (‘j’) key on the computer keyboard. There were two practice problems before the experiment started. The participants did not receive any feedback, neither for practice nor test problems. The problems were presented in random order, and the correct answer appeared equally often on the left and on the right side of the screen. Altogether, the participants took about 15 minutes to finish the experiment.

Results

We used SPSS 23 to analyze the data. In line with data analysis procedures in previous studies on number comparison, we excluded incorrectly solved problems (13.8%, see below) and problems for which the response time deviated more than two standard deviations from the sample mean of the respective problem type (another 3.7% of all problems) for analyzing response times. To compare mean response times between problem types, we used paired samples *t*-tests. As accuracy data were not normally distributed, we used nonparametric tests to compare accuracies between problem types. To analyze distance effects, we ran logistic regression analyses to predict accuracy on the level of individual problems. We used multiple linear regression analyses to predict response times on individual problems as well as sample mean accuracies and sample mean response times.

Differences Depending on the Applicability of Component Strategies

Table 2 provides an overview of accuracies and response times for the problems of the five different types.

Overall, accuracy rate was 86%, which is fairly high but somewhat lower than the academic mathematicians’ accuracy rate on fraction comparison problems (97%) reported by Obersteiner et al. (2013). However, there were substantial differences between problem types. In line with Hypothesis 1, participants were significantly more accurate regarding problems for which assessing the magnitudes of the components was sufficient (CC-CO, CC-IC, No-CC-N), compared to those problems for which this was not the case (No-CC-CO, No-CC-IC) (97% versus 71%), $z = 5.84$, $p < .001$, $r = 0.62$. There was also a significant response time difference between the problems of these two types, $t(44) = 8.56$, $p < .001$, $d = 1.28$, with lower response times for the former compared to the latter type of problems (3604 ms vs. 5540 ms).

Table 2

Means and Standard Deviations of Accuracy Rates and Response Times (on Correctly Solved Items), Depending on Problem Type

		Accuracy (%)		Response Time (ms)	
Type		***M***	***SD***	***M***	***SD***
CC		97.22	3.87	3218	747
	CC-CO	96.19	6.91	2986	788
	CC-IC	98.25	3.78	3451	870
No-CC		78.94	8.07	5620	2085
	No-CC-CO[a]	51.59	28.73	7502	3749
	No-CC-IC	89.84	9.81	5012	1758
	No-CC-N	95.40	7.17	4568	1584
All		86.25	5.40	4659	1485

Note. *M* = Mean, *SD* = standard deviation, CC = common components, No-CC = no common components, CO = congruent, IC = incongruent, N = neutral.

[a]Due to our exclusion criteria (described in the first paragraph of the Results section), the sample size is reduced to 43 for No-CC-CO problems (i.e. for two persons all No-CC-CO were excluded).

Natural Number Bias

Among those problems that had common components, there was no significant difference between congruent and incongruent problems in terms of the accuracy rates, $z = 1.68$, $p = .094$, suggesting that there was no natural number bias for these problem types in terms of accuracy. However, there was a significant and substantial difference between congruent and incongruent problems in terms of response times, $t(44) = 4.32$, $p < .001$, $d = 0.64$, with longer response times for incongruent rather than for congruent problems. Together, these results partly support Hypothesis 2a in the sense that there were “traces” of a natural number bias in terms of response times but not accuracy.

Among those problems that had no common components, the difference between the accuracies of congruent and incongruent problems was large and highly significant, $z = 5.01$, $p < .001$, $r = 0.53$. However, the direction of this difference was counter to our expectation (Hypothesis 2b): while accuracy was very high with respect to incongruent problems (90%), it was only 52% with respect to congruent problems. For the congruent problems, accuracy did not significantly differ from random chance level (50%), $z = 0.38$, $p = .707$. The response times for congruent and incongruent problems revealed the same unexpected difference, namely, significantly longer response times for congruent than for incongruent problems, $t(42) = 4.96$, $p < .001$, $d = 0.76$.

These results suggest that participants not only struggled more with congruent than with incongruent comparison problems, but were actually unable to find successful strategies to solve the congruent comparison problems while being well able to solve the incongruent comparison problems. The analyses of the distance effects will reveal whether the participants relied on assessing the magnitudes of the irrational numbers, or on component strategies alone (which, however, should have resulted in better performance on congruent than incongruent problems).

Distance Effects

To analyze whether the participants engaged in processing the magnitudes of the irrational numbers or of their components, we ran multiple regression analyses, including the distance between the indices, the distance between the radicands, and the distance between the irrational numbers as predictors. For No-CC-CO and No-CC-IC problems, we ran these analyses with both accuracy and—in a separate analysis—response times as the dependent variables. However, for problems of types CC-CO, CC-IC, and No-CC-N, we ran only these analyses which had response times as the dependent variable. The reason was that for the latter problem types, accuracies were extremely high (> 94%), so that linear regression analyses would not yield reliable results.

For modeling accuracy data of No-CC-CO and No-CC-IC problems, we ran the regression analyses twice. First, we ran a linear regression analysis on the sample level, using the sample mean accuracy as the data point for each numerical distance value. Table 3 displays the results of this analysis.

Table 3

Results of the Multiple Linear Regression Analyses With Numerical Distances as Predictors and Sample Mean Accuracy as the Depending Variable, for Each Type of Problems

Type	Predictor	*B*	SE *B*	β	*p*	R^2
No-CC-CO						.67
	Dist_Indices	-0.05	0.02	-.65	.024	
	Dist_Radicands	0.00	0.00	.26	.352	
	Dist_Numbers	0.44	0.19	.51	.037	
No-CC-IC						.54
	Dist_Indices	0.01	0.01	.24	.309	
	Dist_Radicands	-0.00	0.00	-.67	.018	
	Dist_Numbers	-0.03	0.10	-.08	.737	

Note. Dist_Indices = distance between indices, Dist_Radicands = distance between radicands, Dist_Numbers = distance between the irrational numbers, No-CC = no common components, CO = congruent, IC = incongruent.

In a second analysis, we ran a binary logistic regression, using 45 data points (one per participant) for each numerical distance value. Table 4 displays the results of this second analysis.

Although the first regression analysis explained much more variance in the accuracy data than the second one, both types of analyses yielded the same overall pattern of results: for the No-CC-CO problems, the distance between the irrational numbers as well as the distance between the indices were significant predictors of accuracies. This result suggests that for these problems, the participants at least tried to assess the irrational number magnitudes to some extent. However, considering their low accuracies, they were not very successful in doing so. On the contrary, they may have made systematic mistakes on the most difficult problems featuring small numerical distances (accuracies far below 50%). Interestingly, while the distance between the irrational numbers was positively related to accuracies, the distance between indices was negatively related to accuracies. We will discuss an interpretation of this finding in the discussion section below.

Table 4

Results of the Multiple Binary Logistic Regression Analysis With Numerical Distances as Predictors and Accuracy as the Depending Variable, for Each Type of Problems

Type	**Predictor**	***B***	**SE *B***	***p***	**Nagelkerkes R^2**
No-CC-CO					.10
	Dist_Indices	-0.22	0.06	.000	
	Dist_Radicands	0.01	0.01	.121	
	Dist_Numbers	1.91	0.53	.000	
No-CC-IC					.10
	Dist_Indices	0.10	0.07	.138	
	Dist_Radicands	-0.04	0.01	.001	
	Dist_Numbers	-0.20	0.92	.833	

Note. Nagelkerkes R^2 is an estimate of the variance explained by the logistic regression model. Dist_Indices = distance between indices, Dist_Radicands = distance between radicands, Dist_Numbers = distance between the numbers, No-CC = no common components, CO = congruent, IC = incongruent.

Table 5

Results of the Multiple Linear Regression Analysis With Numerical Distances as Predictors and Sample Mean Response Times on Correctly Solved Items as the Depending Variable, for Each Type of Problems

Type	**Predictor**	***B***	**SE *B***	**β**	***p***	***R^2***
CC-CO						.11
	Dist_Radicands	3.85	3.70	.36	.321	
	Dist_Numbers	-88.33	446.72	-.07	.847	
CC-IC						.20
	Dist_Indices	48.38	36.33	.36	.210	
	Dist_Numbers	249.33	311.15	.22	.440	
No-CC-CO						.02
	Dist_Indices	-89.59	199.54	-.19	.663	
	Dist_Radicands	4.20	19.04	.10	.830	
	Dist_Numbers	-163.47	1966.97	-.03	.935	
No-CC-IC						.57
	Dist_Indices	44.08	53.68	.18	.431	
	Dist_Radicands	34.14	12.31	.64	.020	
	Dist_Numbers	-950.43	959.64	-.23	.345	
No-CC-N						.70
	Dist_Indices	-39.74	48.07	-.15	.428	
	Dist_Radicands	20.85	5.65	.68	.004	
	Dist_Numbers	-847.14	528.02	-.29	.140	

Note. Dist_Indices = distance between indices, Dist_radicands = distance between radicands, Dist_Numbers = distance between the numbers, CC = common components, No-CC = no common components, CO = congruent, IC = incongruent, N = neutral.

For the No-CC-IC problems, the distance between the radicands, but no other distance, was a significant predictor of accuracies. This result suggests that the participants relied on comparing the radicands rather than the magnitudes of the irrational numbers. However, we should interpret this result with caution because accuracies were relatively high for the No-CC-IC problems (90%), which, as mentioned earlier, limits the reliability of the linear regression analyses.

For the response times as the dependent variable, we ran the regression analyses separately for all types of comparison problems. Again, we ran each analysis twice. First, we used the sample mean response times for each numerical distance value. Table 5 displays the results of this analysis.

In a second analysis, we used all 45 response times (one per participant) for each numerical distance value. Table 6 displays the results of this second analysis.

Table 6

Results of the Multiple Linear Regression Analysis With Numerical Distances as Predictors and Response Times on Correctly Solved Items as the Depending Variable, for Each Type of Problems

Type	Predictor	*B*	SE *B*	β	*p*	R^2
CC-CO						.00
	Dist_Radicands	3.77	3.92	.05	.337	
	Dist_Numbers	-79.85	471.24	-.01	.866	
CC-IC						.01
	Dist_Indices	48.78	33.82	.06	.150	
	Dist_Numbers	244.85	294.04	.03	.405	
No-CC-CO						.00
	Dist_Indices	38.60	175.69	.02	.826	
	Dist_Radicands	3.20	14.50	.02	.825	
	Dist_Numbers	-154.78	1396.57	-.01	.912	
No-CC-IC						.03
	Dist_Indices	44.49	45.54	.04	.329	
	Dist_Radicands	32.70	11.18	.14	.004	
	Dist_Numbers	-960.80	829.14	-.06	.247	
No-CC-N						.02
	Dist_Indices	-40.51	71.86	-.02	.573	
	Dist_Radicands	21.08	8.44	.11	.013	
	Dist_Numbers	-863.63	775.87	-.05	.266	

Note. Dist_Indices = distance between indices, Dist_Radicands = distance between radicands, Dist_Numbers = distance between the numbers, CC = common components, No-CC = no common components, CO = congruent, IC = incongruent, N = neutral.

While the first type of analysis could explain a substantial portion of the variance in response times, the second type of analysis had poor explanatory power. As with the accuracy data, however, both types of analyses yielded the same pattern of results. The distance between the irrational numbers was not a significant predictor of response time for the problems of any type. This result suggests that the participants did not rely on the irrational number magnitudes to solve the comparison problems of any of the problem types.

For incongruent and neutral problems without common components, the distance between the radicands was a significant predictor of response times, suggesting that the participants relied predominantly on comparing the radicands to solve the comparison problems of these types.

In sum, the analyses of distance effects are largely not in line with Hypothesis 3a, which predicted holistic distance effects for comparison problems without common components. The only holistic distance effect we could detect regarded accuracies on congruent problems without common components. Unexpectedly, we found a small distance effect for the components (indices in case of congruent problems, radicands in case of incongruent and neutral problems) for problems without common components. In line with Hypothesis 3b, we did not find holistic distance effects for those problems for which component strategies are valid. However, we did not find component distance effects for these problems either. We will discuss these results in the next section.

Discussion

We investigated whether mathematically skilled adults are at all able to readily assess magnitudes of irrational numbers presented in symbolic notation. Our study extends previous research that has amply shown that educated adults are able to assess magnitudes of number symbols for integers and fractions.

Participants in our study were able to correctly solve almost all problems for which component strategies were applicable (i.e., problems of type CC-CO, CC-IC, and No-CC-N). This finding suggests that mathematically skilled adults are able to choose very efficient component strategies—and avoid holistic strategies—to solve problems for which these component strategies are valid. In line with that interpretation, the participants were less accurate for comparison problems for which component strategies were not valid (i.e., problems of type No-CC-CO and No-CC-IC). Although we would theoretically expect to find a component distance effect if participants relied on component strategies, we found this effect neither for the radicands nor for the indices for any of the comparison problems with common components. As mentioned earlier, it is likely that although participants actually used component strategies, the component distance effects were too small to be detectable in our sample of mathematically skilled adults. We should also note that due to the restrictions we set for constructing the items, there was not much variation among the indices between comparison problems within each problem type, which makes it less likely to detect a distance effect for these indices.

Surprisingly, while accuracy rates were rather high for incongruent problems without common components, they were at random chance level for congruent problems without common components. The low accuracies for congruent problems suggest that the participants were not able to assess the magnitudes of the irrational numbers. The question is how the participants were able to solve most of the incongruent problems correctly, since there is no straightforward alternative strategy (such as a component strategy) for these problems either. To find an answer to that question, we looked for systematic differences between the congruent and incongruent problems other than congruency. The differences in the numerical distances between the components might explain the results: while the mean numerical distance between indices was similar between congruent (3.00) and incongruent (4.07) problems, the mean numerical distance between radicands was much higher for congruent (61.57) than for incongruent (16.29) problems. This is necessarily the case, given the constraint that the number ranges of the irrational number components as well as the irrational number magnitudes were comparable between problem types. For an irrational number with a larger index to be the larger number (a defining feature of congruent problems), it is necessary to greatly increase the radicand. The participants in our study might have used an approximation strategy: if the distance between the radicands is relatively small, one can assume the radicands are equal and choose the number with the smaller index as the larger number. As even small differences in the indices strongly affect the size of the irrational numbers, comparing the indices yields reliable information in these cases. In fact, choosing the number with the smaller index was (by definition) a successful strategy for all the incongruent problems. The "reverse" distance effects for the radicands of the No-CC-IC problems support our interpretation: The distance between radicands was positively related to response times and slightly negatively related to accuracies (see Tables 3 and 4 for accuracy data, and Tables 5 and 6 for response time data). This means that our participants struggled more with comparison problems when the distance between the radicands was larger than when it was smaller, presumably because it was more difficult for them to use the approximation strategy in the former case than in the latter.

Following this interpretation, we would also expect a distance effect for the indices of the No-CC-IC problems, because after assuming that the radicands are equal, one has to compare the values of the indices. We did not find such a distance effect for the indices, which might, however, be due to the reasons discussed in the Introduction and earlier in this section.

In contrast to the incongruent problems without common components, the supposed approximation strategy (assuming the radicands are equal and choosing the number with the smaller index) was not applicable for the congruent problems without common components, because the distances between the radicands were too great. In fact, in these types of problems, the distances between the radicands were always so great that the number with the smaller index always had the smaller numerical value. The participants in our study might have overestimated the effect of the index on the magnitude of the irrational number, which would explain the difference in accuracies between congruent and incongruent problems without common components. Together, the data suggest that the participants in our study did not find valid strategies other than component strategies (including approximation regarding these components) for comparing the magnitudes of irrational numbers.

The analyses of distance effects support our conclusion that the participants did not engage in processing the magnitudes of the irrational numbers as a whole. We found no holistic distance effects for any type of comparison problems, except for accuracy in the congruent problems without common components. This means that only for this type of problems did the participants engage in assessing the magnitudes of the irrational numbers as a whole to some extent, but often failed to do so successfully. In fact, mean accuracies for many of the congruent problems without common components were far below chance level, suggesting that the participants made systematic errors on some problems. The finding that the distance between indices was negatively related to accuracies in problems of type No-CC-CO might support our interpretation that the participants overestimated the influence of the index on the numerical value of the irrational numbers. While for the No-CC-IC problems, a larger index always meant a smaller number, the reverse was true for the No-CC-CO problems.

The methodology of this study does not allow final conclusions concerning our participants' strategies, because accuracies and response times are an only indirect measure of strategy use. Furthermore, we analyzed data on the level of the whole sample but not on an individual level. Further studies could analyze individual solution strategies to provide more insights into individual participants' reasoning processes. Individual interviews with retrospective verbal reports or think aloud protocols (as described, e.g., in Ericsson & Simon, 1980) could be a suitable method for that purpose. The challenge of assessing individual strategies with these methods is, however, that assessing number magnitudes is a partly automated process, at least regarding the natural number components. Therefore, educated adults might struggle with explaining their solution strategies in every detail. An alternative method that could be used in combination with response times measures and verbal reports is eye tracking. Recently, this method has been used successfully in fraction processing studies (Huber et al., 2014; Ischebeck, Weilharter, & Körner, 2016; Obersteiner & Tumpek, 2016).

We found a natural number bias for problems with common components in terms of response times but not accuracies. This finding is in line with previous research that has detected a natural number bias in comparison problems with fractions (Van Hoof et al., 2013). It is particularly in line with research on academic mathematicians who solved almost all fraction comparison problems correctly but showed a natural number bias in terms of response times in problems with common components (Obersteiner et al., 2013). It seems that the natural number bias is particularly likely to occur in problems in which participants focus strongly on the natural number components (Alibali & Sidney, 2015; Obersteiner, Van Hoof, Verschaffel, & Van Dooren, 2016).

We did not find a natural number bias for problems without common components. On the contrary, the participants were particularly inaccurate and slow in solving congruent problems without common components, but highly accurate and much faster in solving incongruent problems. As discussed above, these differences might be due to other task features, so that we cannot conclude that participants were actually not biased.

This study suggests that even mathematically skilled adults struggle with assessing the magnitudes of irrational numbers. The study thus challenges the idea that quickly assessing numerical magnitudes is an essential part of numerical abilities beyond rational numbers. If quick assessment of magnitudes was essential for being competent with real numbers in general, then people with high mathematical skills should be able to assess magnitudes also of irrational numbers, even if doing so might be more demanding and time-consuming than for rational numbers. Although the participants spent most time on the congruent problems without common components, they performed only at random chance level. It seems that assessing magnitudes for irrational numbers is extremely demanding and cannot be achieved with straightforward strategies.

In view of evidence showing that activating magnitudes of natural numbers is much easier than activating magnitudes of fractions, some researchers have argued that the human cognitive architecture is privileged for processing natural numbers (Feigenson, Dehaene, & Spelke, 2004). Other researchers have contested this view (Huttenlocher, Duffy, & Levine, 2002; Matthews & Chesney, 2015), arguing that adults can process ratios via perceptual routes, and that even young children have a basic understanding of ratios. Processing non-natural numbers might be more challenging just because people have less experience with non-natural numbers, rather than because of a general cognitive disposition. Our study seems to suggest that human cognitive architecture is limited in the sense that it cannot readily assess the magnitudes of irrational number symbols. However, our participants' low performance on the most challenging problems (types No-CC-CO) might be due to the fact that they have encountered irrational numbers too infrequently to be able to quickly assess their magnitudes. It would be interesting to see whether people could improve their ability to assess magnitudes of irrational numbers after specific practice.

In view of its implication for education, this study suggests that assessing magnitudes can be extremely difficult even for mathematically skilled adults. While quickly assessing the magnitudes of number symbols is a feasible goal for specific cases of irrational numbers (e.g., $\sqrt{2}$ or π), it might be impossible to reach such a goal for irrational numbers in general. In fact, assessing magnitudes of irrational numbers in general is not an explicit goal of current mathematics curricula. The boundaries of quick magnitude assessment for irrational numbers, however, do not limit the important goal of students acquiring an understanding of the *concept* of irrational numbers as such, as well as an understanding of the meaning of their symbolic notation. This goal might also include the ability to compare irrational numbers in special cases, where comparison can be based on the irrational number components. Further research could investigate to which extent these goals are reached through current teaching methods.

Notes

i) Note that in academic mathematics, other numbers exist (such as complex numbers), which do not have magnitudes in this sense. That is, there is no order relation for these numbers, so it is not possible to decide which of two numbers is "larger".

ii) Note that there are also rational numbers that have an infinite number of digits after the decimal point (those with recurrent decimals such as 1/3 = 0.333…). However, for these numbers it is possible to notate all recurrent digits (because their number is finite), whereas this is not the case for irrational numbers. This means that irrational numbers cannot be represented in decimal notation.

iii) Note that in the reminder of this article, we use the term "irrational numbers in their symbolic notation" or simply "symbolic notation" to refer to the exact value of irrational numbers presented with the root sign. We do not refer to the decimal notation of numbers, which would only allow approximate representations of irrational number values.

iv) Of course, if one is informed about the category of the comparison problem (congruent or incongruent), one can make valid decisions without assessing the number magnitudes. However, participants are typically *not* informed about the problem category. This was also not the case in our study.

Competing Interests

The authors have declared that no competing interests exist.

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Appendix: List of Number Comparison Problems of Each Type

Common Components Congruent (CC-CO)	Common Components Incongruent (CC-IC)	No Common Components Congruent (No-CC-CO)	No Common Components Incongruent (No-CC-IC)	No Common Components Neutral (No-CC-N)
$\sqrt[3]{14}$ vs. $\sqrt[3]{21}$	$\sqrt[3]{13}$ vs. $\sqrt[6]{13}$	$\sqrt[7]{13}$ vs. $\sqrt[8]{21}$	$\sqrt[7]{21}$ vs. $\sqrt[4]{13}$	$\sqrt[4]{13}$ vs. $\sqrt[5]{12}$
$\sqrt[4]{67}$ vs. $\sqrt[4]{82}$	$\sqrt[8]{26}$ vs. $\sqrt[17]{26}$	$\sqrt[4]{7}$ vs. $\sqrt[6]{97}$	$\sqrt[9]{90}$ vs. $\sqrt[7]{84}$	$\sqrt[5]{23}$ vs. $\sqrt[7]{22}$
$\sqrt[5]{93}$ vs. $\sqrt[5]{37}$	$\sqrt[5]{43}$ vs. $\sqrt[8]{43}$	$\sqrt[11]{56}$ vs. $\sqrt[10]{8}$	$\sqrt[6]{67}$ vs. $\sqrt[9]{74}$	$\sqrt[12]{43}$ vs. $\sqrt[16]{41}$
$\sqrt[6]{15}$ vs. $\sqrt[6]{69}$	$\sqrt[18]{15}$ vs. $\sqrt[23]{15}$	$\sqrt[6]{3}$ vs. $\sqrt[8]{34}$	$\sqrt[10]{15}$ vs. $\sqrt[21]{19}$	$\sqrt[7]{15}$ vs. $\sqrt[6]{19}$
$\sqrt[7]{23}$ vs. $\sqrt[7]{78}$	$\sqrt[8]{52}$ vs. $\sqrt[7]{52}$	$\sqrt[12]{94}$ vs. $\sqrt[11]{38}$	$\sqrt[5]{52}$ vs. $\sqrt[8]{59}$	$\sqrt[8]{52}$ vs. $\sqrt[7]{78}$
$\sqrt[8]{92}$ vs. $\sqrt[8]{88}$	$\sqrt[12]{61}$ vs. $\sqrt[10]{61}$	$\sqrt[15]{26}$ vs. $\sqrt[16]{53}$	$\sqrt[7]{92}$ vs. $\sqrt[5]{55}$	$\sqrt[9]{92}$ vs. $\sqrt[13]{81}$
$\sqrt[9]{12}$ vs. $\sqrt[9]{63}$	$\sqrt[7]{48}$ vs. $\sqrt[5]{48}$	$\sqrt[8]{84}$ vs. $\sqrt[5]{12}$	$\sqrt[6]{15}$ vs. $\sqrt[12]{19}$	$\sqrt[10]{55}$ vs. $\sqrt[9]{70}$
$\sqrt[10]{52}$ vs. $\sqrt[10]{3}$	$\sqrt[10]{34}$ vs. $\sqrt[13]{34}$	$\sqrt[10]{11}$ vs. $\sqrt[12]{86}$	$\sqrt[10]{34}$ vs. $\sqrt[15]{68}$	$\sqrt[7]{76}$ vs. $\sqrt[6]{78}$
$\sqrt[11]{54}$ vs. $\sqrt[11]{21}$	$\sqrt[20]{78}$ vs. $\sqrt[17]{78}$	$\sqrt[3]{4}$ vs. $\sqrt[7]{90}$	$\sqrt[20]{42}$ vs. $\sqrt[12]{29}$	$\sqrt[10]{96}$ vs. $\sqrt[11]{52}$
$\sqrt[12]{87}$ vs. $\sqrt[12]{43}$	$\sqrt[12]{25}$ vs. $\sqrt[14]{25}$	$\sqrt[7]{78}$ vs. $\sqrt[4]{2}$	$\sqrt[14]{63}$ vs. $\sqrt[17]{85}$	$\sqrt[13]{83}$ vs. $\sqrt[14]{72}$
$\sqrt[13]{31}$ vs. $\sqrt[13]{76}$	$\sqrt[15]{74}$ vs. $\sqrt[13]{74}$	$\sqrt[12]{17}$ vs. $\sqrt[15]{72}$	$\sqrt[11]{26}$ vs. $\sqrt[16]{53}$	$\sqrt[14]{48}$ vs. $\sqrt[12]{75}$
$\sqrt[17]{85}$ vs. $\sqrt[17]{65}$	$\sqrt[13]{26}$ vs. $\sqrt[10]{26}$	$\sqrt[19]{77}$ vs. $\sqrt[14]{5}$	$\sqrt[18]{91}$ vs. $\sqrt[16]{73}$	$\sqrt[19]{37}$ vs. $\sqrt[15]{43}$
$\sqrt[13]{55}$ vs. $\sqrt[13]{32}$	$\sqrt[11]{64}$ vs. $\sqrt[14]{64}$	$\sqrt[18]{95}$ vs. $\sqrt[13]{15}$	$\sqrt[10]{44}$ vs. $\sqrt[9]{40}$	$\sqrt[6]{28}$ vs. $\sqrt[4]{30}$
$\sqrt[9]{16}$ vs. $\sqrt[9]{44}$	$\sqrt[13]{33}$ vs. $\sqrt[12]{33}$	$\sqrt[14]{89}$ vs. $\sqrt[5]{3}$	$\sqrt[11]{64}$ vs. $\sqrt[8]{27}$	$\sqrt[10]{34}$ vs. $\sqrt[15]{13}$

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