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Theoretical Contributions

Natural Alternatives to Natural Number: The Case of Ratio

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Abstract

The overwhelming majority of efforts to cultivate early mathematical thinking rely primarily on counting and associated natural number concepts. Unfortunately, natural numbers and discretized thinking do not align well with a large swath of the mathematical concepts we wish for children to learn. This misalignment presents an important impediment to teaching and learning. We suggest that one way to circumvent these pitfalls is to leverage students' non-numerical experiences that can provide intuitive access to foundational mathematical concepts. Specifically, we advocate for explicitly leveraging a) students' perceptually based intuitions about quantity and b) students' reasoning about change and variation, and we address the affordances offered by this approach. We argue that it can support ways of thinking that may at times align better with to-be-learned mathematical ideas, and thus may serve as a productive alternative for particular mathematical concepts when compared to number. We illustrate this argument using the domain of ratio, and we do so from the distinct disciplinary lenses we employ respectively as a cognitive psychologist and as a mathematics education researcher. Finally, we discuss the potential for productive synthesis given the substantial differences in our preferred methods and general epistemologies.

Keywords: number, perception, ratio, rational number

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Introduction: The Need to Consider Alternatives to Natural Number Based Instruction

Mathematical literacy and engagement is essential for participation in modern society, with research showing success in mathematics to be an important determinant of children's later educational, occupational, and even health prospects (Cross, Woods, & Schweingruber, 2009; Garcia-Retamero, Andrade, Sharit, & Ruiz, 2015; Kilpatrick, Swafford, & Findell, 2001; Moses & Cobb, 2001). Generally, the acquisition of a robust sense of number is taken to be an essential first step on the road to mathematical competence (Landerl, Bevan, & Butterworth, 2004; National Mathematics Advisory Panel, 2008). Unfortunately, most efforts to cultivate early number sense rely on count-based, discretized natural number[i] principles – principles that do not align well with a large swath of the mathematical concepts we wish for children to learn. We argue that this misalignment presents an important impediment to teaching and learning. We further suggest that one way to circumvent these pitfalls is to leverage children's non-numerical experiences that can provide intuitive access to foundational mathematical concepts.

In this piece, we explicitly critique the current practice of building nearly all of early mathematics out of counting and associated natural number concepts. Although natural numbers provide an undoubtedly flexible and powerful tool for supporting mathematical thinking, we argue that the practice of introducing almost every fundamental concept in terms of natural numbers often comes with a cost. Namely, to the extent that natural number concepts are misaligned with to-be-learned concepts, they are unlikely to serve as adequate foundations for building understanding of such concepts. There are many ways to conceive of these costs, depending upon one's disciplinary commitments. They can be characterized in terms of negative transfer (Novick, 1988), inhibiting schemes (Streefland, 1991), change-resistance (McNeil, 2014) and epistemological obstacles (Sierpińska, 1987), to name but a few. Our aim is not to provide an exhaustive list of the ways in which these costs have been characterized, but to highlight a family of related costs across disciplines that may apply to overreliance on natural number and to underscore the idea that explicit concern for these costs should inform our theory and practice. Moreover, it is important to consciously recognize

1. that there are often multiple alternative choices, symbolic and nonsymbolic, available for representing early mathematical concepts,
2. that each of these alternatives typically comes with conceptual affordances and constraints, and
3. that number is no exception to points 1 & 2.

Ultimately, we advocate for explicit leveraging of perceptually based intuitions about quantity that may offer different affordances than symbolic numbers. These intuitions can support ways of thinking that may at times align better with to-be-learned mathematical ideas, and thus may serve as a productive alternative for particular mathematical concepts when compared to number.

The Supremacy of Natural Number

Natural number abounds in K-12 mathematics in the U.S., with standard curricular and instructional approaches building early mathematics from counting activities. Students build from counting to whole number addition and subtraction, followed by multiplication (initially as repeated addition) and division (initially as partitioning). Students are then introduced to negative numbers and rational numbers, and only reach ideas about the real numbers in advanced algebra courses in high school (Devlin, 2012; Dougherty, 2008). The primacy of natural number is reflected in policy and standards documents such as the Common Core State Standards for Mathematics (National Governors Association Center/Council of Chief State School Officers, 2010), as well as in the vast majority of elementary and middle school curricula (e.g., Lappan et al., 2006; Putnam, 2003; Thompson & Senk, 2003). This approach obscures the fact that, although numerical representations are convenient vehicles for communicating mathematical concepts, they are seldom the only effective representations available.

Overreliance on number can lead educators and learners to focus on number symbols themselves as opposed to the referent fields to which they should refer (compare with Kaput, Blanton, & Moreno-Armella, 2008). Consider the major Common Core Mathematical Content Standards for Kindergarten (National Governors Association Center/Council of Chief State School Officers, 2010). Although the standards include two sets of ideas that are not exclusively numerical (identify, describe, compare, and compose shapes and describe and compare measurable attributes), the remaining content standards are exclusively numerical. Broadly, the overwhelming majority of the content standards in early elementary are focused on counting and natural number, with many of the ideas about comparison and magnitude subsumed under the counting and cardinality strands. This is despite the fact that the fundamental notion of relative magnitude and the consequent ideas of *greater than* and *less than* pervade everyday life and draw on children's natural experiences. Even though children without numerical training can immediately choose the taller of two people or the larger of two pieces of cake – comparisons involving continuous (Smith, 2012) quantities[ii] – our policies are structured so that early numerical comparison is unnecessarily tied to natural number symbols and the discrete quantity logic they represent. In this case, we have privileged number symbols and relegated the core to-be-learned concept to the background. Moreover, this step is wholly unnecessarily given that children's perceptual abilities provide them intuitive access to spatial relations that easily illustrate comparative magnitude (Brannon, Lutz, & Cordes, 2006; Davydov & Tsvetkovich, 1991; Dougherty, 2008; Dougherty & Slovin, 2004; Feigenson, Dehaene, & Spelke, 2004; Gao, Levine, & Huttenlocher, 2000; Schmittau & Morris, 2004).

This is but one example of what we argue is a more general pattern. In building the majority of early mathematics out of counting and natural number, we have chosen to privilege a particular conceptual basis for much of children's mathematical thinking, and this choice has a number of drawbacks. For instance, prior research has argued convincingly that teaching the multiplication operation primarily in terms of repeated addition of whole numbers can hamper more nuanced understandings of multiplication later on (e.g., Fischbein, Deri, Nello, & Marino, 1985; Harel, Behr, Post, & Lesh, 1994; Smith, 2012; Steffe, 1994; Thompson & Saldanha, 2003; Verschaffel, De Corte, & Van Coillie, 1988). Students persist in conceiving of multiplication as repeated addition, sometimes into secondary school (Anghileri, 1999; Nunes & Bryant, 1996; Siegler, Thompson, & Schneider, 2011), which can lead not only to misconceptions such as "multiplication makes bigger" (Greer, 1988), but also to difficulties conceiving of multiplication with non-natural numbers, envisioning multiplication as continuous scaling, and understanding the inverse relationship between multiplication and division (e.g. Smith, 2012).

Another major deleterious effect of overreliance on counting and natural number is particularly dangerous because it is so counterintuitive: Namely, overreliance on natural number can impoverish the broader concept of number itself. Teaching number as fundamentally tied to counting obscures key continuities shared by all real numbers – for instance, that they can all be ordered and assigned specific locations on number lines (Okamoto & Case, 1996; Siegler et al., 2011). This development of discretized, count-based conceptions of number can lead to deep confusion when natural number symbols are concatenated to represent more advanced numerical concepts such as ratios and associated rational number concepts. This confusion is deeply implicated in prevalent misunderstandings such as the *whole number bias* with fractions, the tendency to use the natural number counting scheme to interpret fractions (e.g., judging 1/8 as larger than 1/6 because 8 is larger than 6; Mack, 1990; Ni & Zhou, 2005). It is important to note that these misunderstandings are not limited to children; even teachers and other highly educated adults fall prey to similar difficulties that can be traced back to overreliance on natural number concepts (Newton, 2008; Reyna & Brainerd, 2008; Stigler, Givvin, & Thompson, 2010). Thus, even with respect to the fundamental concept of number, we argue that number symbols are not always the best place to start. Even in the domain of number, children's perceptions of nonsymbolic spatial figures and their embodied experiences can sometimes provide intuitive semantic access that can be purposefully recruited to make symbols more meaningful (Lewis, Matthews, & Hubbard, 2015).

An Alternate Conceptual Basis

There is evidence that children can build up mathematical thinking in ways that are not always bound to natural number (e.g., Davydov & Tsvetkovich, 1991; Dougherty, Flores, Louis, & Sophian, 2010). This is clear insofar as much of advanced mathematical thinking is supported by various types of visual representations (e.g., Ainsworth 2006; Nistal, Van Dooren, Clarebout, Elen, & Verschaffel, 2009; Rau, 2017) and embodied experiences (Abrahamson, 2012; Alibali & Nathan, 2012; Ellis, Özgür, Kulow, Williams, & Amidon, 2015) that are not numerical in nature. Early mathematics can be built from activities that focus on continuous quantities and consideration of magnitudes qua magnitudes – as amounts of 'stuff' as opposed to a measure that is explicitly numerical. Activities embracing these alternate representations are not merely inferior proxies for numerical representations. In fact, they offer different affordances than what is offered by activities reliant on symbolic numbers, harnessing children's everyday experiences with amount, length, height, movement, volume, and other aspects of their experiential worlds. In the process, such an instructional focus can recruit perceptual apparatuses that evolved over millennia that can provide deeply intuitive semantic access to fundamental mathematical concepts (such as greater than or less than or continuity) independently of numbers.

Indeed, some researchers have proposed that the ability to engage with a generalized sense of magnitude is psychologically prior to the conception of number, laying the foundation for its emergence (e.g., Leibovich, Katzin, Harel, & Henik, 2017; Mix, Huttenlocher, & Levine, 2002; Newcombe, Levine, & Mix, 2015). A larger corpus of psychological literature shows intimate connections between space and number or time and number without committing to which is logically prior (e.g., Dehaene, Dupoux, & Mehler, 1990; Hubbard, Piazza, Pinel, & Dehaene, 2005; Walsh, 2003). Research showing a psychological connection between perceived magnitudes and symbolic numerical magnitudes is perhaps most interesting for what it may suggest about the universe of building blocks available for supporting mathematical thinking (Dehaene et al., 2003; Fischer, Castel, Dodd, & Pratt, 2003; Henik & Tzelgov, 1982; Hubbard et al., 2005; Matthews & Lewis, 2017; Newcombe et al., 2015; Odic, Libertus, Feigenson, & Halberda, 2013; Walsh, 2003). A tight connection between symbolic numerical cognition and nonsymbolic perceptual abilities suggests that nonsymbolic abilities may function as alternative foundations for the same topics we often choose to represent with numbers. It may in fact be that fully embracing the power of the nonsymbolic thinking is the key to maximizing meaningful thinking with mathematical symbols.

In the sections that follow, we will unpack this argument using the domain of ratio, and we will do so from the distinct disciplinary lenses we employ respectively as a cognitive psychologist (Matthews) and as a mathematics education researcher (Ellis). We chose to focus on ratio for two reasons. First, researchers from mathematics education research and from psychology, despite substantial differences in approach, both agree that understanding ratio – and associated rational number concepts – is critical (Anderson, 1969; Beckmann & Izsák, 2015; Behr, Lesh, Post, & Silver, 1983; Lobato, Ellis, & Zbiek, 2010; National Council of Teachers of Mathematics, 2000; National Mathematics Advisory Panel, 2008; Siegler et al., 2012; Siegler, Fazio, Bailey, & Zhou, 2013). Ratio is a foundational topic that supports student's understanding of probability, function, and rates of change that form the basis of algebra, calculus and other higher mathematics. Second, it represents a conceptual domain that continues to provide considerable difficulty for learners despite decades of research dedicated to study of its acquisition (e.g., Anderson, 1969; Cramer & Post, 1993; Ellis, 2013; Latino, 1955; Lesh, Post, & Behr, 1988; Lobato et al., 2010; Novillis, 1976; Smith, 2002).

In the section immediately below, one of us (Matthews) will argue 1) that research from multiple fields suggests that humans have a perceptually-based sensitivity to nonsymbolic ratio that emerges prior to formal instruction and 2) that leveraging these intuitions may prove an effective base for building up formal ratio concepts. In the next, (Ellis) will address how ratio can be developed from magnitude comparison, and will propose a way to foster the development of internalized-ratio from children's perception and mathematization of covarying quantities. In the final section, we will discuss how the divergence in our general methods presents opportunities for synthesis. We ultimately underscore the importance of the convergence in our conclusions given the backdrop of substantial differences in methods and philosophy.

Ratio as Percept (Matthews)

In a recent series of papers, my colleagues and I have argued for a cognitive primitives approach to grounding ratio concepts (Lewis et al., 2015; Matthews & Chesney, 2015; Matthews & Hubbard, 2017; Matthews & Lewis, 2017; Matthews et al., 2016). That is, I have argued that a considerable swath of empirical research suggests that human beings have intuitive, perceptually-based access to primitive ratio concepts when they are instantiated using nonsymbolic graphical representations (Figure 1). My colleagues and I have dubbed this basic perceptual apparatus the Ratio Processing System (RPS) and have called for new research exploring its limits and how it might be used to support mathematical reasoning (Lewis et al., 2015; compare with Jacob, Vallentin, & Nieder, 2012). Unfortunately, I have often been so concerned with explaining my experimental protocols and results that I have failed to give an adequate treatment of what I mean when I use the word 'ratio.' As a result, some colleagues have accused me of playing fast and loose with the terms *ratio*, *fraction*, and *rational number* – often ignoring the distinctions between formal definitions of the terms.

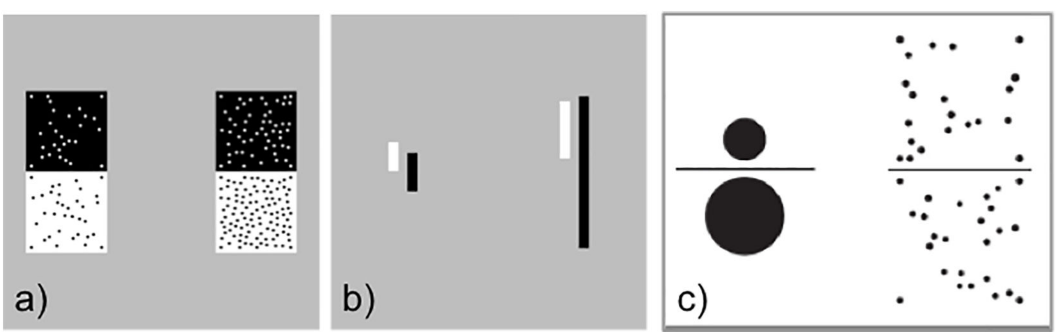

Figure 1. Sample nonsymbolic ratios used in nonsymbolic comparison tasks. Ratios are composed of various type of nonsymbolic quantities, including (a & c) dot arrays, (b) line segments, and (c) circle areas. Note that dot arrays, while countable, are often used in task where the number of dots is high (i.e. > 40) and the time limit is set low (i.e. < 2 seconds) to preclude the possibility of counting. In these instances, dot array numerosity is processed perceptually (for an example, see Matthews & Chesney, 2015).

For my part, I plead guilty to these charges. Indeed, psychologists elide these distinctions every day. Even taking a most cursory glance through a psychology journal, one can find many charts with a *y*-axis marked "proportion correct" to indicate frequency of correct response, which is not true to the mathematically precise definition of proportion as a statement that two ratios or fractions are equal. Devoted formalists shudder! But should we gasp too, when we find the first definition of proportion in the Oxford English Dictionary is "A part, share, or number considered in comparative relation to a whole" ("Proportion," 2016)? What about when the 2nd definition is “the relationship of one thing to another in terms of quantity, size, or number; the ratio”? I cite these definitions not because I take the dictionary to be canon, but because I respect Wittgenstein’s (2001) argument that much of what we know about a word and its associated concepts emerges from the way it is used in everyday language games. The simple fact is that we often use the terms *ratio*, *fraction*, *proportion*, *percentage*, largely interchangeably in everyday language. This is not the result of some linguistic depravity. Rather, it reflects the fact that the language games in which these specific words are caught up are general games involving the broader notions associated with the rational number construct. All share reference to relational quantities that are at root characterized by a concern for how one quantity stands in relation to another. When I use the word ratio, I refer to this generalized sense of a quantity emerging from a comparative relationship between pairs of quantities, and I similarly refer to this generalized sense when I use the word fraction. Thus, for my purposes, the distinctions between the terms are largely immaterial.

My approach is motivated in part by Kieren’s (1976, 1980) influential treatments of rational number concepts. He argued that rational number should be a seen as a mega-concept involving many interwoven subconstructs or interpretations, including:

1. Rational numbers as fractions
2. Rational numbers as decimal fractions
3. Rational numbers as equivalence classes of fractions
4. Rational numbers as ratios of integers
5. Rational numbers as operators
6. Rational numbers as elements of the quotient field
7. Rational numbers as measures or points on a number line.

For Kieren, robust understanding of rational number depends upon having adequate experience with their many interpretations. Although the taxonomies vary somewhat, several prominent researchers have endorsed such a mega-concept view (e.g., Behr et al., 1983; Charalambous & Pitta-Pantazi, 2007; Novillis, 1976), with Ohlsson (1988) arguing that learners’ difficulties with rational numbers stem in large part from “the bewildering array of many related, but only partially overlapping ideas that surround fractions” (p. 53).

Given the complexity of the rational number mega-concept, it is arguably important to find interpretations that square well with children’s intuitions. To that end, my research has focused on promoting the addition of an eighth interpretation to the list above that essentially combines subconstructs #4 and #7: *rational numbers as ratios of nonsymbolic quantities*. This nonsymbolic ratio interpretation is one more or less formulated by Carraher (1996) when he drew the distinction between a ratio of quantities and a ratio of numbers. According to this interpretation, the term ratio need not exclusively refer to a multiplicative relationship between numbers; it can also correspond to a similar relationship between two nonsymbolic quantities, such as the ratio of the lengths of two line segments considered in tandem, e.g., “1/2” instantiated as

Here, each of component line segments is an *extensive* quantity, whose magnitudes can be defined in absolute terms by their lengths considered in isolation. In contrast, the ratio of these nonsymbolic magnitudes is an *intensive* quantity determined by the ratio between them (Lesh, Post, & Behr, 1988).

This nonsymbolic perspective is consistent with Davydov's project to ground fractions instruction in intuitions about measurement (Davydov & Tsvetkovich, 1991; Dougherty, 2008), discussed in more detail by Ellis below. In contrast to Davydov, however, my focus is on nonsymbolic ratio as percept – as an intensive quantity that can be perceived in an *analog* fashion (i.e., in an intuitive, nonverbal, and approximate form). As such, its essence is nonverbal, further eliding the distinctions among ratio, fraction, percentage and other verbally mediated descriptions of rational number concept (see also Van Den Brink & Streefland, 1979).

It is noteworthy that ratios of nonsymbolic quantities are actually much more expansive in reach than the typical conception of ratio introduced with natural numbers. When composed of continuous nonsymbolic quantities such as line segment lengths, the ratios formed extend beyond those corresponding to rational numbers and include all real numbers. Pi, the ratio between a circle's circumference and its diameter, is one such "irrational" ratio that corresponds to a pair of line segments similar to those in Figure 1. The ratio of an equilateral triangle's altitude to one of its sides is another. Indeed, one important aspect of the nonsymbolic ratio is that it can both provide intuitive access to rational numbers and simultaneously provide intuitive access to other classes of numbers, including whole numbers and irrational numbers (see Matthews & Hubbard, 2017; compare with Siegler et al., 2011). Perhaps what is most appealing about the nonsymbolic ratio interpretation is that it does not fundamentally rely upon number symbols to convey the generalized sense of rational number as a quantity emerging from a relationship between pairs of quantity.

The Challenge of Building Ratios From Natural Numbers

It is arguable that the single greatest shortcoming of typical approaches to teaching rational numbers lies in their dependence upon natural number symbols and count-based logic. Although number symbols are typically seen as relatively abstract and flexible, it is arguable that Arabic numerals are rendered concrete relative to many other symbols due to the information that they automatically communicate to experienced learners (e.g., Cohen Kadosh, Lammertyn, & Izard, 2008; Moyer & Landauer, 1967). Thus, repurposing the counting numbers 2 and 3 to represent 2/3, a quantity that cannot be reached by counting, may pose some special problems, precisely because of the concrete nature of our understandings of 2 and 3.

Indeed, much prior research demonstrates that natural number symbols are learned so thoroughly that simply seeing them evokes thoughts about whole number magnitudes. For example, whole number values are encoded so automatically that even children demonstrate Stroop effects whereby they are faster and more accurate in judging that the 7 is physically larger than the 5 in

7 and 5

than they are to indicate that the 5 is physically larger than the 7 in

5 and 7

(Bull & Scerif, 2001; Henik & Tzelgov, 1982; Washburn, 1994). Further, viewing large numbers automatically primes many adults to attend to the right side of space, and viewing small numbers primes attention to the left side of space (Dehaene, Bossini, & Giraux, 1993; Fischer et al., 2003). Moreover, neuroimaging studies show that even passive exposure to symbolic numbers – in contrast to nonnumerical words – activates brain regions used to process discrete sets of objects (e.g., Cantlon et al., 2009; Eger, Sterzer, Russ, Giraud, & Kleinschmidt, 2003; Piazza, Pinel, Le Bihan, & Dehaene, 2007). For decades now, we have known that the ways human beings discriminate between symbolically represented natural numbers parallels the ways that humans perceptually discriminate between magnitudes of different perceptual continua, such as two sounds of different volumes or lights of different intensities (Moyer & Landauer, 1967). This is even true for kindergarten children in familiar number ranges (Sekuler & Mierkiewicz, 1977). These are but a few of the extensively replicated effects that suggest that number symbols are processed rapidly and independently of volition. They bear an informational load that is not easily shed, and this has consequences.

One consequence alluded to above is the inappropriate application of natural number schemas to rational number contexts. For instance, when dealing with rational numbers, learners commonly struggle with the *whole number bias* – a strategic bias to base judgments on a single component of a ratio or fraction rather than on the relationship between the components (DeWolf & Vosniadou, 2015; Mack, 1995; Ni & Zhou, 2005; Obersteiner, Van Dooren, Van Hoof, & Verschaffel, 2013; Pitkethly & Hunting, 1996). For instance, children routinely judge 1/8 to be larger than 1/6, because 6 is larger than 8 (Mack, 1990). Thus, when a nationally representative sample of high school students was asked whether 12/13 + 7/8 was closest to 1, 2, 19, or 21, students were more likely to choose 19 and 21 than 2 (Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1980). That is, many students demonstrate a whole number bias, adding the components of the fractions (i.e., numerators, denominators) in violation of the principles that govern fraction arithmetic.

Although this problem is well known in both mathematics education research and in psychology, the depth of the problem may be underappreciated. Research shows that even mathematicians and undergraduates at highly selective universities are not completely immune to this bias (e.g. DeWolf & Vosniadou, 2011; Obersteiner et al., 2013; Vamvakoussi, Van Dooren, & Verschaffel, 2012). When undergraduate participants are asked to compare symbolic fractions, they tend to be highly accurate overall, but they are significantly slower and less accurate when component based comparisons do not agree with overall magnitude comparisons. These effects are particularly apparent in cases involving double symbolic incongruities – cases in which the larger ratio has both a smaller numerator and smaller denominator compared to the larger – such as 2/9 vs. 1/3. In my own work, I found that this double symbolic incongruity imposed a cost of 400-500ms extra processing time – a massive cost for tasks generally completed in less than 2000 ms. Even though participants were highly accurate overall (mean accuracy > 90%), they were six times more likely to answer incorrectly on these items compared to others (Matthews & Lewis, 2017; see also Ischebeck, Weilharter, & Körner, 2016).

What shows up as relatively small costs in terms of reaction times for the highly educated adult seems to impose a much higher cost on children acquiring rational number knowledge. The prevalence and severity of the costs associated with introducing ratio and fraction with natural number has led some to hypothesize that engagement with whole number schemas *directly inhibit* the acquisition of rational number concepts (e.g., Hartnett & Gelman, 1998; Streefland, 1978; see Pitkethly & Hunting, 1996, for a review). On this view, difficulties with rational numbers stem directly from the entrenchment of discretized natural number logic with which children routinely engage in their formal mathematics experiences (compare with McNeil & Alibali, 2005; McNeil et al., 2012). Because the 2 and 3 in 2/3 are processed involuntarily as natural numbers, it is difficult to get learners to attend to the new number that emerges from the relationship between them given the new notation. Learners cannot see the forest (rational numbers) because of their familiarity with the trees (natural number symbols). To the extent that ratios and fractions must be represented using concatenated and repurposed natural numbers, the difficulty would seem to be nearly inevitable.

In this context little hope is to be gained from any of Kieren's seven interpretations of rational number. Each relies on notations that employ repurposed natural number symbols and are therefore likely to evoke count based logics that are not easily reconciled with the relational logic of rational numbers. However, the eighth subconstruct I have offered – that of the nonsymbolic ratio – is not dependent upon number symbols. If such an interpretation of ratio could be made sensible, it might provide access to important aspects of the rational number mega-concept without falling subject to the pitfalls of symbolic representations.

A Sense of Proportion

Above I alluded to Carraher's (1996) instructive distinction between a ratio of quantities and a ratio of numbers. He summed up what is at stake when pedagogy relies solely on number symbols thusly:

> A narrow mathematical conception disregards the role physical quantity plays in the meaning of the concept and consequently obscures the psychological origins of fractions, for there is little doubt that number concepts, including rational number concepts, are developed through acting and reflecting upon physical quantities. (p. 241)

He proposed that nonsymbolic ratio could go beyond the number line or measurement interpretation (#7), which fundamentally involves a project of mapping nonsymbolic ratios to symbolic numbers (see Chesney & Matthews, 2013). Carraher temporarily shed the connection to number entirely, suggesting that students could come to represent ratios between line segments in a de-arithmetized or non-numerical fashion, without the encumbrance of units (Figure 2; compare with Ellis' paint roller example below). He argued that,

> By suppressing the number line altogether we can work in a non-metric space, with no unit of measure, in which the lengths of line segments are mutually defined according to their relative magnitude…this characteristic makes possible the creation of partially "de-arithmetized" tasks, for which there exist no mere numerical solutions and for which students must reflect upon ratios of quantities. (p. 285)

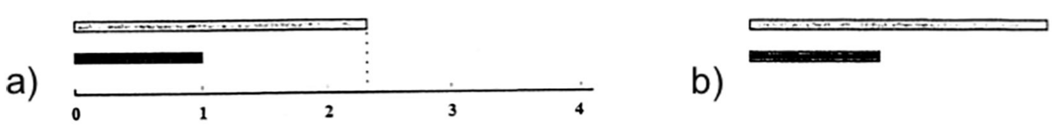

Figure 2. Adapted from Carraher (1993). Panel (a) shows a typical use of nonsymbolic quantities that are paired with natural number labels. By contrast, panel (b) shows the line segments in the same ratio and invites consideration of the relation without reliance on numbers and counting.

This conception presaged recent work that cast number lines as proportions between nonsymbolic ratios and pairs of number symbols (e.g., Barth & Paladino, 2011; Matthews & Hubbard, 2017), focusing on the intuition provided by the nonsymbolic ratio representation. Despite the conviction with which he wrote, Carraher offered limited empirical evidence regarding these "psychological origins." A proper defense would have required showing that human beings do have access to rational number concepts when presented in nonsymbolic form and that this access precedes formal instruction using formal number symbols. Fortunately, the empirical record has confirmed these claims.

A growing number of studies demonstrate that human beings do in fact have a perceptually-based sensitivity to nonsymbolic ratios (Abrahamson, 2012; Duffy, Huttenlocher, & Levine, 2005; Fabbri, Caviola, Tang, Zorzi, & Butterworth, 2012; Jacob & Nieder, 2009; Matthews et al., 2016; McCrink & Wynn, 2007; Sophian, 2000;

Stevens & Galanter, 1957; Vallentin & Nieder, 2008; Yang, Hu, Wu, & Yang, 2015; see Jacob et al., 2012 or Lewis et al., 2015 for reviews). Moreover, this nonsymbolic ratio sensitivity is present early in development, including the years preceding formal instruction on rational number concepts. For example, even 6-month-old infants are sensitive to nonsymbolic ratios when presented as collections of yellow Pac-men and blue power pellets (McCrink & Wynn, 2007, Figure 3a), and preschool-aged children can match spatial ratios such as those shown in Figure 3b (Sophian, 2000). In fact, some research suggests that very young children may at times be more attuned to nonsymbolic ratio than to extensive magnitude; Duffy, Huttenlocher, and Levine (2005) presented 4-yr old children with wooden dowels encased in glass sheaths as shown in Figure 3c and instructed them to pick which of two choices matched an original dowel on length. When presented with a choice that matched the target in terms of dowel length and a target mismatched on length but matched on dowel:case ratio, children picked the proportional match far more often than chance. Thus, not only do young children exhibit ratio sensitivity, but in some cases nonsymbolic ratio is even more salient than a raw match of extensive magnitude. Moreover, they exhibit this competence without reliance upon symbolic numbers.

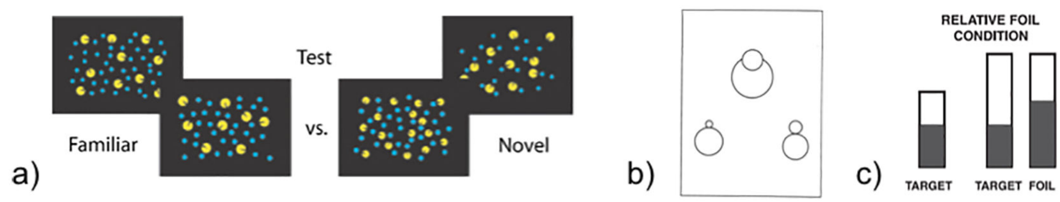

Figure 3. Nonsymbolic ratio stimuli used (a) with nonverbal infants by McCrink and Wynn, 2007, (b) with 4- and 5-year-old children by Sophian (2000), and (c) with 4-year-olds by Duffy, Huttenlocher, and Levine (2005).

Recent research with adults has begun to systematically detail the robust nature of nonsymbolic ratio sensitivity and how it relates to symbolic mathematical abilities (Hansen et al., 2015; Matthews et al., 2016; Möhring, Newcombe, Levine, & Frick, 2016). One of the most important findings is that adults appear to process nonsymbolic ratios perceptually (i.e., without recourse to symbolic numbers). Given their familiarity and accuracy with symbolic fractions, one might expect for university students to convert nonsymbolic ratios into symbolic terms to complete comparison tasks. However, multiple experiments have now found that college students can accurately complete nonsymbolic ratio comparisons significantly *faster* than they complete symbolic ratios (Matthews & Chesney, 2015; Matthews, Lewis, & Hubbard, 2016). This indicates that adults access nonsymbolic ratios without first converting them to symbolic form. Adult performance reveals a perceptual sensitivity – they can feel out ratio values when presented in some nonsymbolic forms (compare with Abrahamson, 2012).

Additionally, research has found that nonsymbolic ratio processing is sometimes automatic (Fabbri et al., 2012; Jacob & Nieder, 2009; Yang et al., 2015). Matthews and Lewis (2017) demonstrated this automatic processing using an adapted size congruity paradigm. As alluded to above, humans are faster and more accurate when choosing the physically larger between

3 and 7

than they are when choosing the physically larger between

3 and 7

This is because numerical magnitudes are read automatically and influence magnitude decisions on the physical dimension. Matthews and Lewis asked undergraduates to compare symbolic numerical ratios while manipulating the physical font ratios, defined as the ratio generated by dividing the numerator font area by the denominator font area. For instance in the left panel of Figure 4, the symbolic ratio 1/4 is written in a larger symbolic ratio than 5/8. For congruent trials, the larger symbolic ratio was presented in a larger font ratio with the smaller symbolic ratio in the smaller font ratio (left panel), and the pattern was reversed for incongruent trials (right panel).

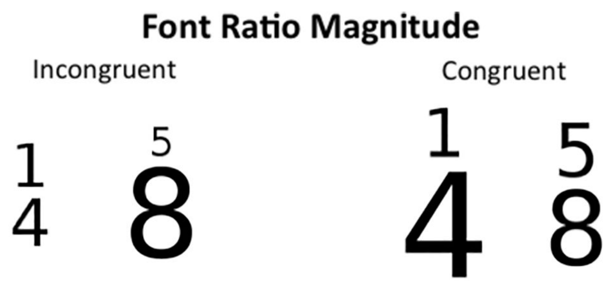

Figure 4. Nonsymbolic font ratio was defined as the ratio of the physical area occupied by the numerator to that occupied by the denominator in fractions used in a comparison task. When the fraction with the larger nonsymbolic font ratio was congruent with the numerical comparison task (i.e., on the same side as the numerically larger fraction), participants were faster and more accurate compared to when nonsymbolic ratio was incongruent (adapted from Matthews & Lewis, 2017).

Participants were faster and more accurate on congruent trials when compared to incongruent trials. Notably, the experimenters never made reference to nonsymbolic ratio for the duration of the experiment. Nevertheless, nonsymbolic ratios were processed automatically, and this nonsymbolic processing influenced symbolic numerical activity. Specifically, larger nonsymbolic ratios were associated with larger symbolic ratio values, and smaller nonsymbolic ratios were associated with smaller symbolic ratios. The research briefly outlined here is but a small swath of the growing body that suggests that humans and other animals (e.g., Harper, 1982; McComb, Packer, & Pusey, 1994; Wilson, Britton, & Franks, 2002) have a basic sensitivity to nonsymbolic ratios and that this sensitivity exists independently of number symbols.

Implications for Education/Training/Didactics

It is perhaps natural to ask, if sensitivity to nonsymbolic ratios is so pervasive and even automatic, why do learners continue to have difficulties with rational number symbols? The answer is a simple: a basic sensitivity does not an elaborated concept make. It is one thing to be perceptually sensitive to nonsymbolic ratio and quite another to have verbally-mediated conceptual knowledge about ratio. It is another thing still to link such conceptual knowledge to repurposed number symbols. On this point, I converge with Ellis in the belief that cultivating formal ratio concepts requires effortful action on the part of the learner. Furthermore, educators will continue to play a crucial role in spurring and supporting this effortful action.

It currently remains the case that most curricula do not attempt to leverage this nonsymbolic capacity. Despite the work cited above to make the case humans can perceive nonsymbolic ratios, very little empirical work in psychology has directly sought to leverage this ability (but see Abrahamson, 2012). On this issue, my work turns from the empirical to the speculative, leaving more open questions than answers. How might we use nonsymbolic ratios as didactic objects (Thompson, 2002) that support mathematical discourse? How might we design perceptual learning modules (Goldstone, Landy, & Son, 2010; Kellman, Massey, & Son, 2010) that use rapid presentation of multiple exemplars to leverage powerful perceptual learning abilities to cultivate conceptual knowledge? Are some nonsymbolic forms more accessible than others? How can we optimally integrate multiple visual forms including nonsymbolic ratios to best illustrate the rational number mega-concept?

Finally, how might we best use the nonsymbolic ratio subconstruct to make symbolic ratios, fractions, and percents more meaningful to learners? Here it is especially important to recognize two conspicuous limitations that nonsymbolic ratios have relative to symbolic numbers: First, despite their intuitive leverage, nonsymbolic ratios still only provide only partial coverage of the rational number mega-concept. If adopted as an eighth construct to Kieren's (1976) list, the other seven are still symbolic in nature. Second, it is clearly the case that learners must learn to engage with symbolically represented rational numbers to be mathematically literate. After all, symbolic representations of rational numbers wield undeniable power and will continue to be the chief currency of advanced mathematics and science. In the final analysis, the chief question is how to make those symbolic representations meaningful instead of seldom understood and often feared. Fortunately, Ellis' section below offers some promising answers to a few of these open questions. Others remain open, and it is our hope that this discussion will help spur future research that will result in more answers.

Ratio as a Mental Operation (Ellis)

I (Ellis) define ratio to be a multiplicative comparison of two quantities. Following Thompson and Thompson (1992; Thompson, 1994), I consider ratio to be constituted in a learner's mental operations grounded in images of multiplicative comparisons and dynamic change. A ratio is not constituted in a situation, in a mathematics problem, or in our environment. Instead, it is formed in one's mental operations when comparing two quantities multiplicatively: "Whether a quantity is a rate or a ratio depends on who is conceiving the situation and upon how he or she happens to be thinking of it" (Thompson, 1993, as cited in Behr, Harel, Post, & Lesh, 1994). One may compare how many times as big one quantity is compared to another, or one may unite two quantities into a composed unit (Lamon, 1994) that can be iterated and partitioned while maintaining the invariant relationship between the linked quantities. Many refer to the latter operation as proto-ratio reasoning (Lesh et al., 1988), but in either case, one must simultaneously attend to two quantities. Further, I propose that one way to leverage students' perceptual, non-symbolic ratio sense is to ground ratio reasoning in images of dynamic change, in which students can consider two quantities whose magnitudes vary together with the anticipation that the invariant relationship does not change (Thompson & Thompson, 1992).

By quantities, I mean mental constructions composed of a person's conception of an object, such as a piece of string, an attribute of the object, such as its length, an appropriate unit or dimension, such as inches, and a process for assigning a numerical value to the attribute (Thompson, 1994, 2011). Like ratios, quantities are not constituted in our environment, but instead are conceptual entities. Many quantities are ones that can be measured directly, such as length. Others, however, must be formed as a comparison. As Matthews noted above when describing ratio as a quantity emerging from a comparative relationship, students can develop a new quantity in relation to one or more already-conceived quantities. This formation does not necessarily have to be multiplicative; for instance, one could compare two quantities additively by considering how much taller one person is than another. However, there is a class of quantities called *intensive quantities* that can be conceived as a multiplicative relation, such as speed or steepness (Nunes, Desli, & Bell, 2003; Piaget & Inhelder, 1975). It is these intensive quantities that are relevant to the formation of the ratio concept.

Thompson (1994) distinguished between the notion of *ratio*, in which a multiplicative relationship is constructed between two static (non-varying) quantities, and *internalized-ratio*, in which the result of a multiplicative comparison remains invariant, but the values of the related quantities vary. This latter notion has some similarities to Kieren's (1976) subconstruct #3, but with special attention to the variation of quantities. Harel et al. (1994) exemplified this distinction in relation to constancy of taste. One can conceive of a ratio such as 3 cups of orange concentrate per 4 cups of water as a comparison of two specific collections, and one can even think about the water as 4/3 as much liquid as the orange concentrate (or the orange concentrate as 3/4 as much liquid as the water). As an internalized-ratio, however, one conceives of the same statement as a representation of all possible ratios between two collections of orange juice and water, where the specific values of the collections vary multiplicatively. Further, each of those ratios will represent the same phenomenon, that of how orangey the mixture will taste. In the following paragraphs I propose a set of instructional principles for developing ratio and internalized-ratio that draw not on natural number, but rather on students' abilities to perceive, isolate, and compare continuous quantities.

In order to form an intensive quantity and conceive of an internalized-ratio, one must form a *multiplicative object* (Moore, Paoletti, & Musgrave, 2013; Saldanha & Thompson, 1998; Thompson, 2011), a conceptual uniting of two quantities so that they are held in the mind simultaneously. The formation of such an object means that one can choose to track either quantity's value "with the immediate, explicit, and persistent realization that, at every moment, the other quantity also has a value" (Saldanha & Thompson, 1998, p. 299), because they are coupled. A person forms a multiplicative object from two quantities as a result of mentally uniting their attributes to make a new conceptual object (Thompson & Carlson, 2017). However, the formation of a multiplicative object is not trivial. Research on children's understanding of ratio thoroughly documents the use of additive rather than multiplicative strategies (e.g., Hart, 1988; Inhelder & Piaget, 1958; Karplus et al., 1983), as well as the phenomenon of reasoning with only one quantity at a time rather than simultaneously attending to both (e.g., Noelting, 1980). These challenges pose obstacles for developing internalized-ratio, a key conceptual advance for understanding proportion, rate, and function.

Building Ratio From Measurement of Magnitudes

Are there ways to better support children's attention to quantities in ratio development? Above, Matthews thoroughly documented the conceptual drawbacks associated with building ratio from natural number symbols, but is there an alternate instructional route? Indeed there is precedent for developing early mathematical ideas by building on children's natural predilection for noticing and comparing quantities they encounter in their lives. In particular, consider the Elkonyn-Davydov curricular approach, a first through third grade Russian curriculum derived from the work of Vygotsky and Leontiev (Davydov, Gorbov, Mukulina, Savelyeva, & Tabachnikova, 1999). Researchers in the United States have subsequently developed a U.S. version of the Elkonyn-Davydov curriculum called *Measure Up*, and have studied its effects on children's algebraic reasoning (Dougherty et al., 2010). These approaches rely on the notion that situating early mathematics in the comparison of non-symbolic magnitudes creates favorable conditions for the future formation of the abstraction of arithmetic and algebraic relationships (Davydov & Tsvetkovich, 1991). The Russian group and their U.S. counterparts begin with non-symbolic continuous attributes, such as length, height, and volume, to model real number properties and operations. This approach is consistent with the notion that numbers are rooted epigenetically in reasoning about quantities and their relationships (Thompson & Carlson, 2017). As Devlin (2012) argued, working with non-symbolic attributes can lead to abstraction; one must think about the quantity itself, and its relation to other quantities, without putting a number to it. In this manner the curriculum does not develop algebraic structure as a generalization of number or the activity of counting discrete objects. Instead, it develops algebraic structure from relationships between quantities (Schmittau & Morris, 2004).

The quantities children attend to in the Elkonyn-Davydov and *Measure Up* curricula are not built from counting; instead, children begin by *comparing* properties of objects and *representing* such comparisons without measuring them. Children come to school already attending to quantities and magnitudes in their everyday activity. They make comparisons about who has more or less, who is taller than whom, and they perceive amounts of extensive and intensive quantities. In the first grade, children are asked to identify which quantities are less than, greater than, or equal to other quantities and then to express these comparisons without assigning numbers to the attributes under comparison. Schmittau (2004) argued that when children do not have access to numerical examples, they must rely on a theoretical understanding of the relationships in question. Thus children's nascent understandings of algebraic relationships are grounded in their attention to quantities in their surrounding environments, actions on those quantities, and models of relationships between those quantities.

With fractions and ratios, researchers using variants of the Elkonyn-Davydov curriculum have worked with children to develop ratio reasoning from a basis of measuring continuous quantities rather than counting and partitioning (Morris, 2000; Simon & Placa, 2012). Students are tasked with objectives such as selecting a length of string that is the same length as a model on the other side of the room, but the model cannot be moved. Thus they learn that one can compare two quantities indirectly through the use of a third quantity that can be compared to both of the original quantities. Students develop the concept of measure, a unit of measure, and a unit, making distinctions among the quantity being measured (such as length), the measure (such as a piece of string), and the number used to designate the relationship between the two. Thus the concept of unit is developed from the use of a chosen measure, and ad hoc units are present early on. This approach offers a contrast to the typical treatment of measurement, in which children begin first with counting activities, working with collections of discrete objects in which each whole object takes on the unit of *one*. Studies examining students' development of algebraic reasoning suggest that this approach, in contrast to a natural-number based approach, may create a more flexible foundation for abstracting algebraic relationships (e.g., Dougherty & Venenciano, 2007; Dougherty et al., 2010).

Within the Elkonyn-Davydov curriculum and its counterparts, the conceptual origin of ratio is a relationship between two magnitudes. This is similar to Matthews' description of ratio as a quantity that emerges from a comparative relationship, with one important difference: Here, after engaging in activities of noticing and comparing, children eventually quantify magnitudes with numbers. Critically, this quantification is built on a foundation that draws on children's proclivities to notice and assess magnitudes. In contrast, typical approaches introduce fraction and ratio through partitioning, relying on a model of division that is inherently discrete. When working with a continuous attribute, such as length, children can choose a measure that is any size relative to the attribute being measured. The attribute being measured may not be an exact multiple of the measure, which affords attention to the relationship between the change of numbers and the change of measures. Further, it moves children out of the world of natural numbers and, at least mathematically, situates ratio within the reals in the sense that neither the value of the measure nor the measured must necessarily be rational.

The above approach to ratio builds on children's abilities to notice and perceive non-symbolic extensive quantities such as length or volume. One could also extend this approach to connect to children's perception of non-symbolic ratios, or intensive quantities. For instance, as Matthews discussed, it may be fruitful to build instruction on having children assess differences in ratio magnitude based on a perceptual comparison of quantities (e.g., Abrahamson, 2012; McCrink & Wynn, 2007; Sophian, 2000). One could then develop a series of activities in which children have to determine whether their perceptual assessments are valid. I suggest it would be most effective to do this not with same-quantity comparisons in ratio perception (such as ratios composed of pairs of line segments shown in Figures 1b and 2), but with ratios representing different-quantity comparisons. For instance, consider the cases of speed, composed of time and distance, or of steepness, composed of height and length. Given two subjects walking at constant speeds, children can easily determine who is walking faster (Ellis, 2007a; Lobato & Siebert, 2002). Likewise, given two ramps, children can easily determine which is steeper (Lobato & Thanheiser, 2002). The question is one of how to support students' activities of appropriately mathematizing those perceptual experiences. How can you measure which person is faster? What quantities make up the determination of speed, and how do you compare those quantities? A great deal of work in mathematics education has occurred precisely in this area, studying ways to situate ratio development in the mathematization of comparing intensive quantities (e.g., Harel, Behr, Post, & Lesh, 1994; Kaput & Maxwell-West, 1994; Lobato & Siebert, 2002; Lobato & Thanheiser, 2002; Simon & Blume, 1994; Thompson, 1994; Thompson & Thompson, 1992). However, this approach stands counter to the curricular dominance of natural number-based tasks. More work is needed to better understand how to leverage both students' perceptual experiences and quantitative reasoning abilities and how to support teachers in implementing such tasks. I propose that a particularly fruitful direction is in the development of internalized-ratio rooted in the comparison of *varying* quantities.

Building Internalized-Ratio From Covariation

Thompson and Carlson (2017) suggest that the operation of dis-embedding an image of one quantity varying (such as distance accumulating) from an image of another quantity varying (such as time accumulating) not only marks the beginning of covariation, but is also at the root of the construction of intensive quantities. By covariation, I refer to the mental activity of holding in mind individual quantities' magnitudes varying, and then conceptualizing two or more quantities as varying simultaneously (Saldanha & Thompson, 1998; Thompson & Carlson, 2017). As researchers or instructors we can develop situations for students that introduce what we conceive of to be constant rates, such as distance and time accumulating together, but it remains an open question whether our students will conceive of those situations covariationally. How to support students' conceptions of those situations through the mental activity of covariation in order to build up internalized-ratio – and, more generally, function – is a persistent question that guides my research program.

Covariation is critical for developing an understanding of internalized-ratio, proportionality, and more broadly the concept of function as a foundation for algebra and calculus. Thompson and Carlson (2017) argue that variational and covariational reasoning are, at their core, fundamental to students' mathematical development. Students can and do reason statically about quantities, conceiving of them as unchanging. But it is also possible for students to develop a conception of a situation that relies on a quantitative structure that supports an image of multiple quantities' values varying in relation to that structure. Saldanha and Thompson (1998) suggest that such images of covariation are, at least in part, developmental. Children may initially coordinate the values of two quantities by thinking of one in isolation, and then the other, and then the first again, and so forth. Subsequently, images of covariation appear to rely on an image of one quantity varying that contains an implicit image of another quantity embedded within it (e.g., Ellis, Özgür, Kulow, Williams, & Amidon, 2015; Keene, 2007; Lobato et al., 2012). Students must then undergo a conceptual shift in which they dis-embed the image of the implicit quantity from the other quantity, becoming able to explicitly attend to the variation of both.

How can we foster the development of these quantitative images to support the construction of internalized-ratio? A key component of such an endeavor is situating students' mathematical activity, initially, in contexts that are not reliant on a basis of counting, number, or even measurement. Removing students' abilities to count and therefore discretize continuous, changing, or unspecified quantities could potentially encourage the construction of multiplicative objects. In fact, I propose that it may be helpful to introduce a variety of tasks that encourage covariational reasoning in order to foster such a Way of Thinking (Harel, 2008) as a broad conceptual foundation before then introducing students to particular quantitative relationships, such as those represented by constant ratios. Here I will discuss two specific tasks that my project team and I have used in a small-scale teaching experiment (Ellis, 2016). Neither task is specific to building ratio, but both were engineered to foster covariational reasoning, which then served as a conceptual foundation for approaching later ratio tasks.

The first task is the triangle/square task (Figure 5).[iii] Figure 5 contains four images that are screen captures of a movie shown to students. In the movie, the point P travels along the perimeter of a square and sweeps out a green triangle as it travels. It begins at the leftmost corner of the square, A. It travels along the bottom of the square from A to B, sweeping out the triangle as it travels (Figures 5a and 5b). The point P continues traveling from B to C, and from C to D in turn, continuing to sweep out the triangle (Figures 5c & 5d). The point P then continues down the left side of the square, from D back to A. During this last portion of P's journey, there is no triangle.

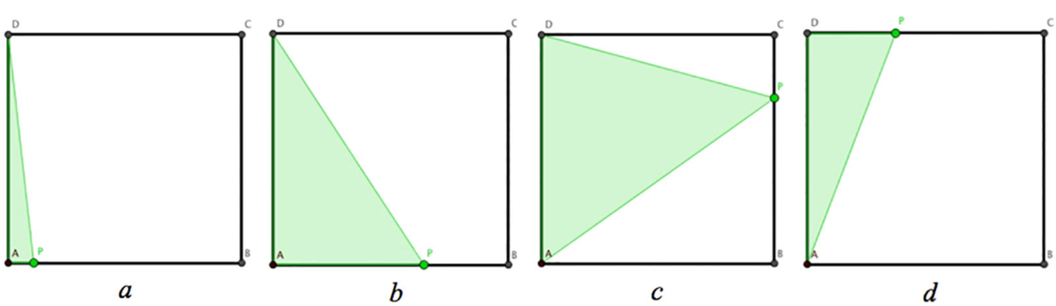

Figure 5. Four screen captures of a movie of a point sweeping out a triangle.

In a teaching experiment (Cobb & Steffe, 1983; Steffe & Thompson, 2000) with two 7th-grade participants, Olivia and Wesley[iv], I provided the students with the triangle/square movie and asked them to think about how the area of the triangle changes compared to the distance P accumulates as it travels around the perimeter of the square. It is possible for one to invent measures for the dimensions of the square and actually calculate areas and distances, which would eliminate the need to simultaneously attend to the two varying quantities. However, given my knowledge of Olivia and Wesley's mathematical backgrounds and our prior experiences, I was confident that they would not do so. Indeed, both students attended to the changing area and the distance traveled without making any measurements. Olivia and Wesley produced similar graphs comparing area with distance traveled (see Figure 6 for Olivia's graph). Olivia explained her thinking as follows:

> So on the first increase, I thought that this, it gets from smaller to bigger [pointing from A to B], and then here was sort of this [pointing along segment BC] because I thought that it would stay about the same area. That's just kind of how I saw it. And then it gets smaller [pointing from C to D]. And then it stays the same at [points along segment DA], I see nothing.

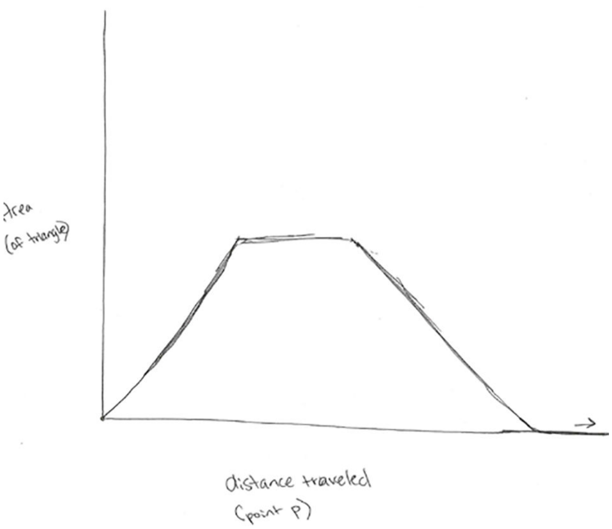

Figure 6. Olivia's graph comparing the triangle's area to distance traveled

Explaining why she made the angled portions of her graph straight rather than curved, Olivia said:

> I think because like here [points to A], it has no area, and here [drags finger towards B] it has a little bit more. Like if you went up by increments here, maybe it would be, like, doubled, and so it kind of goes up. Like kind of what we were talking about before. Here it would be a whole bunch of little lines because it just kind of keeps going, but if it stopped, I imagine it going steadily up, like slowly getting to half.

Olivia's construction of the graph relied on a non-numerical image of comparisons as she thought about the triangle's area value *changing*; she could think about the triangle's area increasing as P traveled, she could think about it decreasing, and she could think about it remaining the same. Olivia later provided a geometric argument for why the triangle's area must remain the same as P traveled from B to C, but she could not explain why she thought the rate of change of the triangle's area from A to B and from C to D would be constant. Her assumption that the triangle's area changed at a constant rate relative to the distance P traveled, which happens to be correct, was likely based on perception. In addition, because the value of P's "distance traveled" increased monotonically, the triangle/square task was an easier covariation task than many tasks of a similar nature. Simultaneously attending to the triangle's area compared to the changing distance between P and A (around the perimeter of the square, not as the crow flies) was a more conceptually complex task, as evidenced by Olivia's initial and corrected graphs (Figure 7). Here, Olivia initially constructed the triangle, but on further reflection realized that the graph would actually be a parallelogram, which included the final leg of the journey from point D to point A on the *x*-axis.

Figure 7. Olivia's graph comparing the triangle's area to the distance between P and A.

The triangle/square task encouraged the students to attend to two quantities that changed together. It is one of a class of tasks that can potentially foster covariation by asking students to think about the changing values of two quantities simultaneously, particularly when it is difficult or impossible to measure (and therefore discretize and calculate with) the quantities' magnitudes. The second task is the Gainesville Task, which we have borrowed from Moore's (2016) study in which he addressed the distinction of graphs representing figurative versus operative thought (Figure 8). With this task, students must simultaneously attend to the car's distance from Gainesville and distance from Athens as it travels from Athens to Tampa and back. The Gainesville task can be particularly effective in encouraging covariation because neither quantity is monotonically increasing; thus, one cannot easily keep one of the quantities implicit in imagination. Further, neither quantity is perceivable in the sense that students cannot see the relevant magnitudes, which removes measuring as an option. They must instead mentally construct images of the quantities changing (contrast this with Carraher's (1996) line segment task, shown in Figure 2).

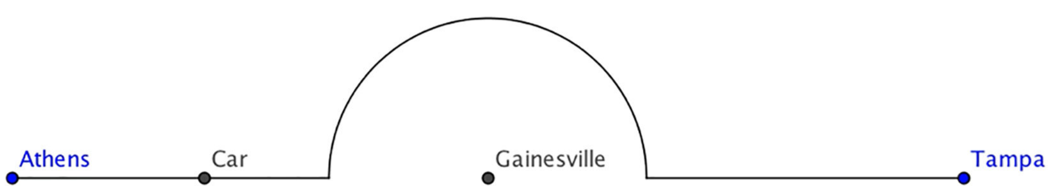

Figure 8. Moore's (2016) Gainesville task.

Wesley's graph (Figure 9) showed that he initially approached the graph in a traditional manner, thinking of both quantities beginning from zero, but he quickly realized his error:

> So I started by going like this (points to the line departing from the origin), but I had to get closer to Gainesville so I'd have to be going backwards so I decided to start here (points to the non-origin point on the *x*-axis). So then I'm getting closer to Gainesville and farther away from Athens. So it goes higher. And then here on this part (pointing to the vertical line segment), you stay the same amount

away from Gainesville but you keep getting farther away from Athens. And then on this part (pointing to the angled line segment at the top of the graph) you get farther away from both.

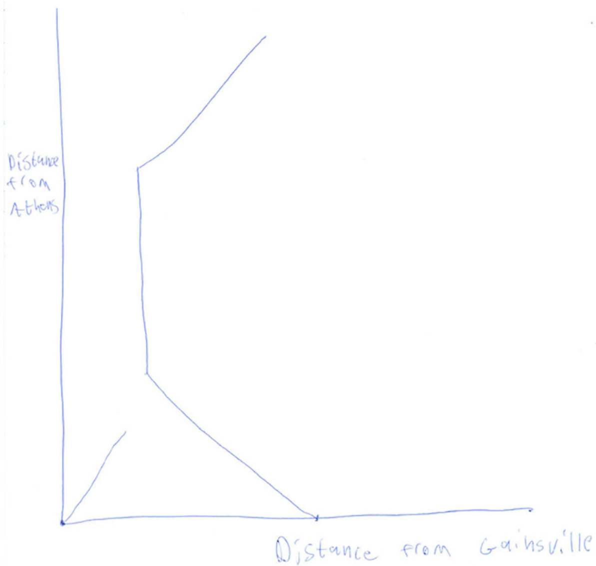

Figure 9. Wesley's graph comparing the distance from Athens to the distance from Gainesville

Building on a foundation of covariational reasoning, one can then shift to tasks that foster building internalized-ratio as an invariant multiplicative comparison in which the related quantities co-vary. For instance, in the same teaching experiment (Ellis, 2016), I introduced tasks in which students compared the changing area of a rectangle compared to its distance swept as its length increased. Here, students familiarized themselves with the context of sweeping out a rectangle of paint with a paint roller of unspecified height (Figure 10), and then watched videos of paint rollers sweeping out rectangles. Finally, the students were asked to think about how the area of the rectangle would change compared to the length swept.

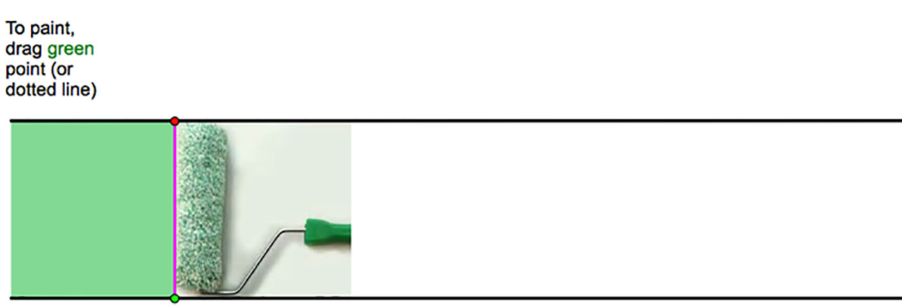

Figure 10. Paint roller task.

By this point, the students were accustomed to attending to how two quantities varied together, and both students graphed the area of the rectangle compared to the length swept as a straight line. Olivia explained, "I sort of pictured it in my head…for every length that you've pulled, it should be the same amount of area." Olivia could imagine the change in area for an unspecified length (the "length that you've pulled"), but unitized the length pulled to imagine how much area would be added for each same-increment amount of length. Wesley, in contrast, invented specific values: "I decided I would think the height would be 1 meter…so like if you drag it out

1 meter, so that the length is 1 meter and the height is 1 meter, and then to find the area you times the length by the height which is one times one, is 1 square meter actually." Before introducing specific lengths (which the students called heights) for the paint rollers, I asked them to think about the rate of change of the area of a rectangle made by a taller paint roller versus a shorter paint roller. Both students graphed steeper lines for the taller paint roller (see Figure 11 for Olivia's graph).

Figure 11. Olivia's graph for the area painted versus the length swept for a shorter and taller paint roller.

Wesley compared the second unspecified height to his original 1-meter paint roller, and explained, "Because since it's, the height is bigger than 1 meter now, then for every length that you pull it out one meter, it gets more area." Similar to Olivia's unspecified length increments, Wesley thought about 1-meter increments and then could compare the amount of area being added for each rectangle while holding the length increment steady. Olivia compared the area swept (which she referred to as the height of the wall) to the slope, stating, "The steeper it is, the larger height of the wall it is."

The ratio of area to length swept is a multiplicative invariant that depends on the height of the paint roller. Thanks to multiple experiences with covariation tasks before encountering the paint roller task, Wesley and Olivia could not only observe the quantities area and length varying together, but could also explicitly think about both quantities changing. It was only then that they were introduced to tasks with specific numbers. When considering paint rollers of specific heights, such as 10 inches or 16.25 inches, the students could construct ratios as representations of various snapshots in the paint-rolling journey. For instance, consider a paint roller that has swept a length of 20 inches to produce an area of 160 square inches. One could imagine the ratio of 160 square inches: 20 inches as having swept out 8 inches in area for each inch in length swept, but one could also conceive of the 160: 20 ratio as one of an equivalence class of ratios, each representing a different point in the paint-rolling journey for an 8-inch paint roller. Further, the height of the paint roller does not have to be a natural number, nor does the amount of inches swept. When one's reasoning is grounded in images of continuously changing quantities rather than discrete collections of sets, non-natural numbers do not pose a problem. Students also have a conceptual foundation for making sense of how changing the magnitude of one quantity or the other can change the value of the ratio.

Another example in which students can reason about how changing initial quantities affects the value of a ratio is from a different covariation context, this time with two spinning gears (Ellis, 2007a). In this teaching experiment students developed ratios by comparing the simultaneous rotations of two connected gears, a large gear and a small gear. Any pair of rotations represented a ratio that the students could consider as one snapshot of a dynamic situation in which the two gears continuously rotated together. If the sizes of the two gears remained the same, then any given rotation pair must represent the same ratio. Further, students could think about how the gear ratio would change if one gear became twice as small, or twice as large. Here as in the rectangle context, non-natural numbers did not pose difficulties for the students.

In the gear scenario, the students constructed a ratio as an *emergent quantity* (Ellis, 2007b). The emergent quantity was gear ratio, or the relative size of one gear to the other as measured in a multiplicative comparison of teeth. Harel and colleagues (1994) identified a key conception in developing internalized-ratio to be an understanding of the (multiplicative) constancy of the relationship. Similarly, Simon and Blume (1994) introduced the term ratio-as-measure, to describe the notion of a ratio as an appropriate measure of a given attribute. One who conceives a ratio as an emergent quantity, that is, an attribute with a meaningful quantitative structure in its own right, can anticipate individual values of the initial quantities changing while the emergent quantity remains invariant. This way of reasoning can potentially be difficult if one has built up a conception of ratio as a static comparison of discrete sets. In contrast, there is some evidence that students develop more flexible ratio reasoning when they can conceive of an emergent quantity with contextual meaning, such as the height of a paint roller, gear ratio, speed, constancy of taste, or steepness (Ellis, 2007b). Further, constructing emergent-quantity ratios that represent conceptions of covariation better situates students to make sense of function, particularly in terms of understanding constant rates of change in algebra, varying rates of change in advanced algebra courses and beyond, and instantaneous rates of change in calculus.

Matthews asked how one might use nonsymbolic ratios as didactic objects to support mathematical discourse in the learning of symbolic ratio. One potentially fruitful avenue is to leverage students' perception and mathematization of co-varying quantities. In the above examples, symbolic number was introduced late, after students had constructed an invariant relationship between two quantities as they varied. Significant time and attention was devoted to supporting students' construction of multiplicative objects, and students had opportunities to learn how to make sense of two quantities varying together before they measured anything. Numbers then became expressions of magnitudes for the students, and thus were imbued with quantitative meaning as representations of their intended referents. In contrast, beginning ratio instruction with natural numbers situates ratio in the discrete world, as a static comparison of sets. It is difficult to then bootstrap up to an equivalence class conception of ratio in which each component value can take on any real number. Situating ratio instruction instead in contexts of variation and change can leverage students' experiences with quantity and variation in order to foster a more robust understanding.

Discussion

Building early mathematics concepts exclusively on natural number and count-based logic comes at a cost. We have presented arguments that converge on this conclusion, despite substantial differences in our preferred methods, contrasts in the types of warrants we choose for our claims, and disagreements in our general epistemologies. Specifically, we have each argued that natural number concepts do not neatly align with the structure of many foundational mathematical concepts, using ratio as a case for illustration. This misalignment can impoverish mathematical thinking, standing as an obstacle to conceptual development. Consistent with this critique, we have also sought to highlight the importance of embracing alternative methods of introducing early mathematics concepts. Despite the present curricular primacy of natural number, other methods of representation often exist for building early mathematics concepts. In particular, we have focused on nonsymbolic representations, arguing that they a) may better leverage children's intuitions built from everyday experiences and b) may better align with the structure of some to-be-learned concepts.

At this point, however, we have yet to give an extensive treatment to the sometimes dramatic differences in our theoretical perspectives and how a theoretical synthesis or interdisciplinary research project might be possible. In what follows, we first attend explicitly to our points of divergence and the implications of this divergence for our research questions and preferred methods. Next, we offer a detailed exposition of the potential for synthesis in the face of this divergence. Finally, we conclude with a general consideration of several ways in which collaborations among researchers with divergent theoretical commitments can help enrich inquiry.

Points of Divergence

1. Epistemological Divergence and Implications for Defining Ratio

In the test domain of ratio, we have pressed our arguments from significantly different epistemological stances. In fact, our perspectives differ in fundamental ways that we have not fully reconciled. These differences have implications for the definition of ratio itself: In particular, are ratios 'constructed', or are they 'things in the world'?

Matthews' argument is built on the notion that humans come equipped with a ratio processing system – a primitive perceptual apparatus that is sensitive to ratio magnitudes. For Matthews, a full accounting of mathematical knowledge involves both metacognitive and associative mechanisms (e.g., Crowley, Shrager, & Siegler, 1997). That is, mathematical thinking clearly involves explicitly recognized conceptual knowledge, procedural knowledge, and procedural flexibility (Schneider, Rittle-Johnson, & Star, 2011). However, beyond this, it also involves associative mechanisms that operate below the level of consciousness. Accordingly, mathematical cognition is implicated in ineffable production rules (Anderson, 1993/2014), and probabilistically activated representations and strategies (e.g., Siegler, 1996). Moreover, it is integrally tied to perceptual learning mechanisms that are at one and the same time both simple and capable of extracting deep conceptual structures from a vast array of information (Goldstone & Barsalou, 1998; Kellman et al., 2010). In sum, this conception of the human learner conceives of the learner as a computer – but not as a computer built from silicon and wires. Instead, this computer is built from billions of neurons and trillions of synapses whereby memories, knowledge, and even emotions are conceived of as patterns of neural activation. For some, this account may seem de-humanizing, reducing human cognition to input-output patterns. For others – including Matthews – it represents a productive world of nearly infinite possibilities, one that can support theorizing about the constraints and potential of human cognition while allowing for incredible flexibility along the way.

From this perspective, human brains did not evolve to do mathematics. Instead, cultural inventions such as mathematics and reading co-opt pre-existing brain systems to support new competencies (see Dehaene & Cohen, 2007). Thus, it is important to see ratio as a salient feature given in the world, something to be observed, like color or brightness (Jacob et al., 2012). Matthews considers ratio as a phenomenon that exists independently of humans and can be passively observed by primitive cognitive endowments that we share with other animals like chimpanzees (Wilson, Britton, & Franks, 2002), lions (McComb, Packer, & Pusey, 1994), and even ducks (Harper, 1982). It is a feature for which our perceptual apparatus has evolved sensitivity over millions of years, and it seems only natural that we should try to leverage this capacity. On this view, ratio is also a fuzzy concept that so seamlessly blends into other rational number concepts, such as fraction or percentage, that they are effectively proxies for one another.

For Ellis, mathematical knowledge develops as part of a process in which children gradually construct and then experience a reality as external to themselves (Piaget, 1971). This theory of knowing breaks with convention in that knowledge does not necessarily reflect an objective ontological reality, but instead is considered an ordering and organization of a world constituted by one's experience (Von Glasersfeld, 1984). This stance does not reject the existence of an objective reality and should not be confused with solipsism (for a detailed account of this argument, see Von Glasersfeld, 1974). Rather, it posits that there is no way to obtain confirmation that our knowledge is an accurate reflection of reality. In order to claim any sense of invariance between what we perceive and what exists external to us, a comparison would have to be made, and we do not have access to such a comparison. Consequently, the idea of a match with reality is replaced with that of fitness. Our knowledge is viable if it stands up to experience, enables us to make predictions, and allows us to bring about certain phenomena (Von Glasersfeld, 1984). It is an instrument of adaptation that enables us to avoid perturbations and contradictions, but adaptive fit is not interpreted as a homomorphism.

Where the realist believes mental constructs to be a replica of independently existing structures, Ellis takes these structures to be constituted by people's activity of coordination (Von Glasersfeld, 1995). A knower's conceptual *and perceptual* activity is constitutive; in other words, our representations of our perceived environments are always the results of our own cognitive activity. Percepts – even color and brightness – are no exception to this. On this view, ratio is not something that exists external to humans that they passively perceive. It is a mental operation coordinated from prior actions and operations (Piaget, 2001), which necessarily entails effortful activity supported by meaningful instruction. This conception of ratio is clearly defined and distinct from the concept of fraction.

2. Implications for Research

These differences in perspective naturally shape the research questions we each pose and the methods we use to investigate them. Matthews begins by asking, how do human brains – these computers composed of neurons – come to fluently understand concepts that they clearly did not evolve to support? From this perspective, it is natural to focus on primitive sensitivity to nonsymbolic perceptual ratios that extends across species and to ask how this sensitivity can be *exapted* for human use. Exaptation, a phrase coined by Stephen J. Gould, refers to traits or features that were not built by natural selection for their current use, but have affordances that organisms leverage for new functions (Gould & Vrba, 1982). Matthews begins by asking about abilities and constraints of the cognitive system – conceived of as a thing in the world that evolved to perceive things in the world – and thinks of education as a process that exapts these basic abilities to support competence with cultural inventions such as mathematics and reading (see also Dehaene & Cohen, 2007; Feigenson, Dehaene, & Spelke, 2004; Gelman & Meck, 1983).

On this view, the first step in theorizing about potential for learning is sometimes to identify and detail these basic primitive abilities (e.g., sensitivity to nonsymbolic perceptual ratios). Methods for exploring these abilities include using various tasks to measure the acuity with which people at various points in development can discriminate among nonsymbolic ratios that serve as analogs to specific numerical ratios. These tasks may involve timed magnitude comparisons (e.g., Matthews & Chesney, 2015), choosing which of two alternative choices matches a target (e.g., Boyer & Levine, 2012; Vallentin & Nieder, 2010), or having participants make estimates using either symbolic numbers or by producing equivalent nonsymbolic ratios in alternative formats (e.g., Meert, Grégoire, Seron, & Noël, 2012). Moreover, these tasks can be adapted to measure the extent to which sensitivity to nonsymbolic ratios automatically affects other magnitude comparisons (e.g., Fabbri, Caviola, Tang, Zorzi, & Butterworth, 2012, Matthews & Lewis, 2017). Finally, correlational studies (e.g., Fazio, Bailey, Thompson, & Siegler, 2014; Jordan, Resnick, Rodrigues, Hansen, & Dyson, 2017; Matthews, Lewis, & Hubbard, 2016) can help reveal whether and how perceptually-based nonsymbolic ratio abilities are related to competence with symbolic ratios and other rational numbers. Each of these methods relies on a large number of data points to support statistical inference about the extent to which samples of people at different ages demonstrate sensitivity to nonsymbolic perceptual ratios in the world.

In contrast, rather than taking ratio as given and studying how well students perceive, leverage, or manipulate ratios and their associated concepts (or how they can be trained to improve in these tasks), Ellis investigates a different set of questions: What do students construct as ratio? What is the structure of the epistemic student's (Steffe & Norton, 2014) mathematical world? In building models of student thinking, what are the possible operations and images entailed in developing a productive ratio concept? How can instruction foster that conceptual development?

The methods appropriate for addressing the above questions include the design and analysis of a) written assessments (e.g., Cooper et al., 2011; Knuth et al., 2012; Lockwood, Ellis, & Knuth, 2013), b) structured or semi-structured clinical interviews (e.g., Ellis & Grinstead, 2008; Lockwood, Ellis, & Lynch, 2016), and c) teaching experiments (e.g., Ellis et al., 2016; Ellis et al., 2015; Ellis, 2011). Written assessments can provide data on students' current ways of operating, and can also be fruitful for identifying common strategies and errors. An advantage of assessment data is that, unlike clinical interviews or teaching experiments, written assessments afford the analysis of large data sets, lending statistical validity to findings. In order to address the finer-grained questions about students' conceptual operations, Ellis relies on clinical interviews and teaching experiments. Clinical interviews (Bernard, 1988; Clement, 2000; Schoenfeld, 1985) involve designing mathematical tasks, asking students to solve those tasks, and eliciting students' ideas with appropriate extension questions. Designed to help the researcher understand and characterize students' concepts of ratio, clinical interviews provide a snapshot of the participants' current ways of operating. A semi-structured interview affords the researcher a fair degree of freedom in following a student's line of reasoning, improvising new tasks, and creating and testing hypotheses about the student's understanding on the spot (Ginsburg, 1997).

In order to investigate the nature of student learning and the development of concepts over time, some mathematics educators rely on the teaching experiment (Cobb & Steffe, 1983; Steffe & Thompson, 2000). Teaching experiments often leverage initial findings from written assessments and clinical interviews in order to develop a tentative progression of tasks. Researchers then work with a small number of participants (although this number can range from 1 to an entire classroom of students) over an extended number of sessions in order to develop models of student thinking and learning through teaching interactions. The teaching experiment method demands a flexibility requiring any initial sequence of tasks to serve only as a rough model of instruction. During and after each session, researchers will engage in iterative cycles of teaching actions, formative assessment and model building of students' thinking, and revision of future tasks and invention of new tasks on an ongoing basis. This flexibility enables the ongoing development, testing, and revision of hypotheses about students' conceptions throughout the data collection process.

Towards Synthesis – From Conflict to Complementarity

These differences noted, we argue that the very divergence in our questions and methods holds much productive potential. Multi-method techniques have long been theorized to be good ways to build richer understandings of complex constructs (e.g., Brewer & Hunter, 1989; Johnson & Onwuegbuzie, 2004). Unfortunately, research methods are in large part dictated by the theoretical commitments of investigators. Thus, to date, the rich classroom experiments preferred by Ellis and the basic cognitive tasks privileged by Matthews have de facto been cast as incompatible or mutually exclusive methodological choices. This need not be the case. Indeed, perhaps the most powerful experience in the current collaboration has been that it required each of us to consider seriously the merits in the other's method. It would be inaccurate to describe the result as a theoretical sea-change for either, but it has certainly piqued interest, prompted theoretical elaboration and led to new questions. We briefly adumbrate a couple of ways that the collaboration has led to some convergence below.

Convergence on ratio. On first glance, the differences in the definition of ratio that we each advance might appear to be one of our most intractable disagreements. After all, one of us has explicitly argued that ratios *are* things in the world and the other has argued that they *are not*. However, there is much more common ground here than initially meets the eye. On the one hand, Matthews now acknowledges the value of thinking about ratio as human-constructed multiplicative mathematical object. Indeed, this recognition is at the core of his call for methods to "mathematize" ratio. On the other hand, Ellis now acknowledges the value of attempting to leverage perceptual experience. In fact, this recognition is part and parcel of her pedagogy described above, which relies heavily on using students' perceptions of dynamic motion to promote reflection on the nature of covariation. In the final analysis, the disagreement is less about the foundations of each other's arguments and more about the proper use of the term 'ratio'.

This appears to be a classic case of what Wittgenstein (1953/2001) referred to as "bumps that the understanding has got by running its head up against the limits of language" (p. 41). In the respective sections above, for Ellis the term ratio was technical and precise, referring to a very specific sort of mathematical object, whereas for Matthews the term referred to something broader and fuzzier that is even somewhat accessible to non-human animals. We have realized much of the tension in our approaches is resolved if we distinguish between two concepts that we might refer to as "perceived ratio," which is chiefly nonsymbolic and approximate, and "mathematical ratio," a precise, language mediated concept. In coming to understand that we were referring to two related but somewhat distinct constructs, a common question emerged: How can we help learners leverage perceived ratio to build mathematical ratio? Several potential strands of research questions naturally branch off from this larger guiding question, questions that our divergent methodologies might combine to explore in complementary ways. Below we will elaborate on one such question and allude to a few more.

How might we leverage perceptual learning using nonsymbolic perceived ratio to help provide a foundation for constructing mathematical ratio? Researchers interested in perceptual learning have long argued that perception is not simply synonymous with low-level sensation, but is highly selective and can be a source of complex and abstract understandings (Barsalou, 2008; Gibson, 1979/2014; Goldstone & Barsalou, 1998; Goldstone, Landy, & Son, 2010; Kellman, Massey, & Son, 2010). It would be interesting to use perceptual training from cognitive science to facilitate intuitive understanding of perceptual ratio and to adopt techniques from mathematics education research to scaffold construction of mathematical ratio from naïve foundations in perceptual ratio.

For example, consider the dynamic perceptual ratio matching task shown in Figure 12. In this example, students are presented with a target nonsymbolic ratio composed of line segments on the left and encouraged to manipulate the length of the orange line segments to create a match on the right. Upon completion, the student is given immediate feedback regarding a properly constructed match. Because the target and the response vary with regard to the absolute size and alignment of component line segments, there are relational features of the stimuli that must covary in specific ways (i.e., the orange:blue length ratio) to maintain equivalence of the perceptual ratios involved. Over a multitude of trials, such practice might help tune students' perceptually-based intuitions to the relations among those core features. When coupled with appropriate prompts, the use of the displays could facilitate rich conversations that can help make explicit the intuitions behind the perceived equivalence. This in turn might help transform a naïve sensitivity to equivalence among perceptual ratio stimuli into a conscious appreciation of a ratio as a mathematical object.

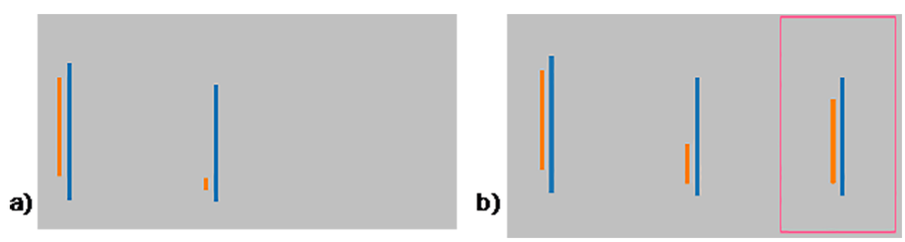

Figure 12. Sequential screenshots from a dynamic perceptual ratio matching task. (a) Participants are initially presented with a target ratio composed of line segments on the left side of the screen along with an incomplete ratio to the right. Participants are asked to adjust the components of the incomplete ratio so that the two match. In this case participants are asked to adjust the length of the orange bar to try to make the orange:blue line ratios match. (b) Once the participant submits a response, the computer gives feedback on what the correct answer would have been (farthest right). In this case, the red box lets the participant know that their submission (center) is quite far from the correct value. Note that for these ratios, the jitter between orange and blue lines is an irrelevant dimension that ensures participants cannot make a match by simple scaling of an identical figure.

The above-described matching tasks would enable us to not only identify how well students perceive equivalent ratios, but also potentially to foster better attention to and conscious appreciation of perceptual intuitions over time. Interviewing a subset of the participants would provide additional data about the particular relational features students consciously attend to, the manner in which they attend to the simultaneous changes in the lengths of the provided lines, and the concepts on which they draw when constructing ratio matches. The combined data from the dynamic perceptual ratio matching tasks and clinical interviews would then inform the development of a hypothetical learning trajectory (Clements & Sarama, 2004; Simon, 1995) for building internalized-ratio through covariation. This hypothetical trajectory would include a set of potential tasks and activities and associated conceptual stages through which we predict students would progress. We could then test this designed intervention in a series of teaching experiments, relying on quantitative pre- and post-assessments to triangulate findings from the qualitative data.

This is but one example of many questions of mutual interest in this domain that might be more profitably investigated by the cooperation between our disciplines than if we continued to conduct research in our respective silos. Others include: Are some quantities more optimal than others for helping learners develop mathematical ratio through observing covariaton? How do students' perceptual intuitions about ratio affect the quantities they attend to when constructing mathematical ratio? Can we measure how integrating measures from mathematics education researchers with those from psychologists might provide more resolution for measuring rational number knowledge than either used in isolation?

This is not to suggest that integrating methods will completely bridge the theoretical and methodological divides between us. In the final analysis, some of our prior theoretical commitments may be irreconcilable, and the key is accepting this fact and moving on in spite of such differences. Our fields stand to profit in several ways from embracing methodological pluralism and forging on with such integrative projects. By incorporating different methods, we generate enriched data sets that we can each use within our preferred analytics. That is, independent of deep theoretical shifts, the presence of more abundant data obtained by more varied measures can present a larger space for inquiry for our preferred analytics. Second, in sharing data and perspective, we create real reflexive opportunities that can lead to genuine shifts in theoretical perspective. Surely the reach of such shifts will be limited somewhat by the depths of our commitments, but the potential remains significant despite these constraints. Third, although our methodological orthodoxies may constrain our abilities to push for a deep synthesis, those steering conventions need not limit our students in the same ways. By offering emerging researchers the opportunities to analyze phenomena of interest through different lenses, we create the possibility that our trainees may produce a more profound synthesis.

A Final Note on the Collaboration

With this project, we have not resolved all of our disagreements, nor do we think this would have been a realistic goal. Our epistemological orientations are far enough apart that hope of substantial reconciliation is probably chimerical – and we feel this is not unusual (but also not inevitable) for cognitive psychologists and mathematics education researchers. Still, we have found substantial common ground on a topic that is of great importance for each of us and for our respective fields. In the process, we have each gained considerably by engaging with the other's mode of thought. We have grown, and our work is richer as a result. It is our sincere hope that this can be a model for collaborations between mathematics education researchers and psychologists more generally. Each has its unique insights, and each has its blind spots. By working across boundaries, and learning to bracket our differences, it may be that focusing on our points of convergence can lead to new productive points of inquiry. In this fashion, we might hope to more exhaustively investigate constructs of mutual interest and to push knowledge further.

Notes

i) The natural numbers are typically defined either as the set of non-negative integers (e.g., 0, 1, 2, 3…) or the set of positive integers (e.g., 1, 2, 3,…). For the purposes of this paper, we mean the set of positive integers when we refer to natural numbers.

ii) Smith (2012) defines continuous quantities to be unitary objects that can be measured, whereas discrete quantities are collections of countable objects.

iii) Thanks to Brandon Singleton for developing the triangle/square task.

iv) Olivia and Wesley are gender-preserving pseudonyms.

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Competing Interests

The authors have declared that no competing interests exist.

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