Journal of Numerical Cognition

PsychOpen GOLD

Editorial

Check for updates

Mathematical Flexibility: Theoretical, Methodological, and Educational Considerations

Marian Hickendorff[1], Jake McMullen[2], Lieven Verschaffel[3]

[1] *Institute of Education and Child Studies, Leiden University, Leiden, The Netherlands.* **[2]** *Department of Teacher Education, University of Turku, Turku, Finland.* **[3]** *Center for Instructional Psychology and Technology, University of Leuven, Leuven, Belgium.*

Journal of Numerical Cognition, 2022, Vol. 8(3), 326–334, https://doi.org/10.5964/jnc.10085

Published (VoR): 2022-11-16

Corresponding Author: Marian Hickendorff, Institute of Education and Child Studies, Leiden University, PO Box 9555, 2300 RB Leiden, The Netherlands. E-mail: hickendorff@fsw.leidenuniv.nl

Related: This article is part of the JNC Special Issue “Mathematical Flexibility”, Guest Editors: Marian Hickendorff, Jake McMullen, & Lieven Verschaffel, Journal of Numerical Cognition, 8(3), https://doi.org/10.5964/jnc.v8i3

Abstract

The current paper presents an introduction to a special issue focusing on mathematical flexibility, which is an important aspect of mathematical thinking and a cherished, but capricious, outcome of mathematics education. Mathematical flexibility involves the flexible, creative, meaningful, and innovative use of mathematical concepts, relations, representations, and strategies. In this introduction we discuss the most relevant theoretical, methodological, and educational considerations related to mathematical flexibility, which form the background of the empirical studies presented in the special issue. Collectively, these studies provide a broader understanding of the mathematical flexibility, its subcomponents, influences, and malleability.

Keywords

flexibility, adaptivity, solution strategies, mathematical representations, conceptual knowledge, domain-general skills, domain-specific skills

Mathematical flexibility is an important aspect of mathematical competence (Newton et al., 2020) and a cherished, but capricious, outcome of mathematics instruction. It involves the flexible, creative, meaningful, and innovative use of mathematical concepts, numerical relations, and mathematical representations and strategies and therefore requires a blend of procedural and conceptual knowledge, of skill and understanding (Baroody, 2003; Schneider et al., 2011), together with various motivational and affective dispositions (Krutetskii, 1976). It is an indicator of deeper mathematical understanding and key for the development of later mathematical competence and success (Kieran, 1992; Krutetskii, 1976; Star et al., 2015). It is therefore not surprising that it has increasingly received attention from cognitive and educational psychologists as well as mathematics educators (Heinze, Star, et al., 2009).

This special issue presents current research aimed at advancing our scientific insight into mathematical flexibility. In this introduction, we review some major theoretical, methodological, and educational issues in the domain of mathematical flexibility and forecast the empirical studies in this special issue, that each address some combination of these three aspects of the study of mathematical flexibility

Theoretical Issues

In our consideration of mathematical flexibility we focus on two main theoretical issues: first, the variety of definitions of mathematical flexibility, and second, the relations between mathematical flexibility and cognitive and non-cognitive factors.

Definitions

Mathematical flexibility is defined in various ways in the literature, ranging from rather narrow to quite broad. A common conceptualization focuses on the use of various strategies to solve mathematical problems. Within this strategy-focused conceptualization there are two different perspectives on what constitutes mathematical flexibility. In the first, narrow, perspective, flexibility only refers to the knowledge and use of various strategies (Heinze, Marschick, et al., 2009; Hickendorff, 2020; Schneider et al., 2011; Verschaffel et al., 2009). In the other perspective, mathematical flexibility is conceptualized broader, as an overarching construct involving both knowledge and use of various strategies as well as the ability to apply the most appropriate strategy on a given problem (Blöte et al., 2001; Newton et al., 2020; Star & Rittle-Johnson, 2008; Star & Seifert, 2006).

However, this focus on strategy use does not present a full picture of what mathematical flexibility entails (Threlfall, 2009; Verschaffel et al., 2009). For instance, the knowledge and ability to use multiple mathematical representations (including graphical, tabular, verbal, and symbolic ones) and to adaptively switch between them within and between problem situations is another well-investigated aspect of mathematical flexibility (Acevedo Nistal et al., 2009; Heinze, Star, et al., 2009). Furthermore, a broader conceptualization of mathematical flexibility introduces the notion that individuals can also show flexibility in understanding of and reasoning about both concepts and procedures in connection with each other (Baroody, 2003), for instance with the well-connected knowledge of numerical characteristics and relations, termed adaptive number knowledge (McMullen et al., 2017).

In this special issue we did not start from an *a priori* definition of mathematical flexibility, but instead aimed to include a versatile repertoire of research using various conceptualizations of the term. All studies give insights into certain aspects of mathematical flexibility, which we conceived of as the flexible, creative, meaningful, and innovative use of mathematical concepts, relations, strategies, and representations, including how task, subject, and/or contextual factors may affect students' flexibility.

Relations With Mathematical Flexibility

Theoretical models and research findings suggest that mathematical flexibility is related to several cognitive and non-cognitive factors.

Within the cognitive factors, a distinction can be made between domain-specific and domain-general factors. One relevant domain-specific factor is conceptual understanding, and the question is how strategic flexibility is related to students' understanding of the underlying mathematical principles (Baroody et al., 2009; Peters et al., 2012; Rittle-Johnson et al., 2001). Some authors have already explored this relationship in the case of flexible switching between counting on from the first or from the second addend in problems such as $2 + 7 = \_$ and $6 + 2 = \_$ on the one hand, and their understanding of the commutativity principle of addition on the other (Baroody & Gannon, 1984). For other, more complex, mathematical principles such as the inversion principle (i.e. $n + m - m = n$; Rasmussen et al., 2003), the relationship with strategy flexibility has so far remained largely unexplored. Evidence does suggest that algebra conceptual knowledge predicts later strategic flexibility (Schneider et al., 2011).

Another relevant domain-specific factor is overall mathematical performance. Although several studies showed that students who show more mathematical flexibility in a particular task domain tend to have higher levels of overall mathematical ability (Hickendorff et al., 2010, 2018; Newton et al., 2020; Torbeyns et al., 2006, 2017), there is not much strong empirical evidence that being flexible when solving a particular mathematical task actually leads to significantly higher performance on that task. Recently, Van Der Auwera, Torbeyns, and colleagues (2022) found that elementary school children who flexibly switched between direct subtraction and subtraction-by-addition when solving multi-digit mental subtraction problems performed better on this mental subtraction task both for accuracy and speed than children who systematically applied one of these strategies. However, Torbeyns and colleagues (2017) found no performance advantage of children who flexibly switched to the compensation strategy on problems such a 573 − 299 = _, compared to children consistently using the same strategy. Fazio and colleagues (2016) found that high-performing university students could be successful on fraction comparison tasks, even when using questionable strategies. However, low performers were not able to reliably switch to more advantageous strategies when given the option. In sum, high mathematical performance may be a predictor of mathematically flexible behavior, but research findings are inconclusive regarding whether being flexible on a task results in higher performance on that task. This issue is taken up in the current special issue, as Garcia-Coppersmith and Star (2022) investigate this complex relationship between strategy selection and accurate execution.

Besides domain-specific factors, there are also domain-general cognitive factors that are likely related to mathematical flexibility. There is ample research evidence that executive functions (shifting, inhibition, and working memory) are important for developing mathematical proficiency (Cragg & Gilmore, 2014). In the current special issue, Eaves and colleagues (2022) address the role of domain-general (working memory, inhibition, and switching) and domain-specific (calculation skill and understanding of the order of operations) skills in the identification of the associativity shortcut on problems such as 16 + 38 − 35 = _. Furthermore, Van Der Auwera, De Smedt, and colleagues (2022) examined the role of domain-general skills (updating, inhibition, and shifting) in individual differences in strategy efficiency and adaptivity.

Turning to the relation with non-cognitive factors, we first consider the role of gender. Not all studies report gender differences in students' flexibility, but if they do it is in favor of boys (Hickendorff et al., 2010, 2018; Timmermans et al., 2007). Girls are found to be relatively more inclined to use the standard strategies taught in the mathematics curriculum, whereas boys are relatively more inclined to use innovative strategies and representations that may be more efficient (e.g., task-specific shortcut strategies; Hickendorff, 2018) but may also be more risky (e.g., using mental computation on problems that put heavy demands on working memory such as 864: 36 = _; Hickendorff et al., 2010). Multiple studies in the current special issue address the role of gender.

Affective factors may also be related to mathematical flexibility. One affective factor is mathematics anxiety, which is known to be negatively related to mathematics performance on routine tasks (Dowker et al., 2016). However, little is known about the relation between mathematics anxiety and tasks that require flexible and adaptive behavior for which students can rely less on their routine behavior. In the current special issue, Halme and colleagues (2022) investigate the relation between mathematics anxiety on the one hand and routine and adaptive number knowledge on the other. While various scholars have made a plea to pay more attention to the role the aesthetic pleasure derived from the beauty or elegance of mathematics (e.g., Krutetskii, 1976), these kinds of affects have been investigated rarely in relation to students' mathematical flexibility.

Despite their importance (Verschaffel et al., 2009), little research has focused on the influence of contextual factors on flexible choice of strategies or representations in the domain of mathematics. One of the first studies showing that children switch both strategies and representations according to the socio-cultural context was done in Brazilian street vendors (Nunes Carraher et al., 1985). Depending on the context – selling on the street or doing mathematics in school – the children not only switched strategies (from mental computation to algorithmic procedures) but also representations (from internal to external representations). Reviews of the (limited) research addressing the influence of the socio-cultural context on the kind of strategies or representations students choose to apply on a given mathematical task and the flexibility of these choices concluded that there is at least limited evidence for contextual influences, but that more research should be conducted to examine these factors futher (Ellis, 1997; Verschaffel et al., 2009). Recently, Hickendorff (2020) manipulated the context in which students had to solve multidigit subtraction problems: a mental or a written computation approach, or or free choice between the two. Allowing or requiring written computation decreased the likelihood that students adaptively switched strategies, which suggests that also the context within a classroom (the socio-mathematical norms) affects mathematically flexible behavior.

Methodological Issues

An important question, of course, is how to assess mathematical flexibility. Given the variety of conceptualizations of the construct (Newton et al., 2020; Verschaffel et al., 2009), it is not surprising that different assessments have been used.

Since solution strategies are crucial for many conceptualizations of mathematical flexibility, the identification of the strategies students use is often pivotal to its measurement. A common approach is to let students solve a set of tasks, for instance arithmetic problems or algebra exercises, and infer their solution strategies from their written work (e.g., Hickendorff, 2018), their (retrospective) verbal reports (e.g., Torbeyns et al., 2015), and/or their reaction- time data (e.g., Luwel et al., 2001; Peters et al., 2010). This gives insight in the repertoire of strategies students use. An important consideration is that there may be a gap between *knowing* and *using* different strategies (Blöte et al., 2001; Newton et al., 2020; Star & Rittle-Johnson, 2008; Xu et al., 2017). Strategy knowledge may be assessed by prompting students to solve the same problem more than once with different strategies or by asking them to evaluate worked-out examples of different strategies (Rittle-Johnson & Star, 2007).

Moreover, if it is possible to designate certain strategies as innovative or efficient shortcut strategies for particular tasks (e.g., using the compensation strategy on problems such as 837 − 399 = _), charting to what extent and on what tasks students use these strategies gives information about whether students make adaptive strategy choices with respect to task characteristics (Heinze, Marschick, et al., 2009; Heinze et al., 2018). In this special issue, Obersteiner and colleagues (2022) show that to assess flexible and adaptive strategy use in fraction comparison tasks, the features of the tasks need to be carefully constructed.

However, strategies that are efficient from the perspective of the task (having the fewest and/or easiest computation steps) are not necessiraly efficient for individual students, since it requires understanding of relations between numbers and between operations and/or good mastery of certain (sub)procedures that are part of that strategy (Hickendorff, 2018; Newton et al., 2020; Torbeyns et al., 2009). This issue can be accommodated by using the choice/no-choice methodology, a comprehensive method to assess various aspects of strategy use (Siegler & Lemaire, 1997). A major strength of this method is that it is possible to compare the efficiency of different strategies in an unbiased way using the no-choice conditions, wherein participants have to use a given strategy to solve all problems. That information can be used to identify which strategy is most efficient for an individual on a particular problem, and these efficiency data can be used to assess the extent to which individuals are adaptive to their own strategy efficiency data, or in other words, to what extent they use the strategy that for them works best on a particular problem. There are, however, also disadvantages to the choice/no-choice method (Luwel et al., 2009; Threlfall, 2009). Most importantly, it has to be decided in advance which two, or sometimes three, strategies are focal. As a result the free choice condition is actually a restricted choice condition and may not fully reflect students' strategy repertoire and flexibility. In the current special issue, Hickendorff (2022) implemented an innovative, personalized version of the choice/no-choice method that does not require restriction of strategy choice, enabling to compare the task-specific strategy to students' own preferential strategy. Furthermore, she distinguished between a practical component (what students do) and a potential component (what students know).

The issue of appropriateness of a particular choice is also relevant in the history of research on representational flexibility. Initially, methodological approaches in this domain mostly conceptualized and operationalized representational flexibility as choosing the representation that provides the best match to the characteristics of the task to be solved. But researchers gradually also started to broaden their methodological scope, for instance, by including data-gathering and/or data-analytic methods that allow the inclusion of students' abilities, preferences, and affects interacting with these representations, as well as the context in which such interaction takes place (Acevedo Nistal et al., 2009; Heinze, Star, et al., 2009).

Beyond examinations of strategy or representational choice, other methods have been developed for examining other aspects of mathematical flexibility. For example, in examining aspects of students' abilities that support flexible strategy choice, McMullen and colleagues (2017) introduced a measure of adaptive number knowledge, the arithmetic sentence production task, that assesses their ability to create arithmetic sentences from a given set of numbers (e.g. 2, 4, 8, 12, and 32) that equal a target number (e.g. 16). The quantity and complexity of responses is argued to measure the richness of a students' knowledge of numerical characteristics and arithmetic relations.

In sum, there is wide variety in the methodologies used to measure specific aspects of mathematical flexibility, each of which provides a specific lense with which the topic is studied.

Educational Issues

Although mathematical flexibility is deemed an important objective of mathematics education curricula, fostering it can be challenging. Mathematical flexibility is something that develops over time, not easily nor quickly (Verschaffel et al., 2011), nor equally in all students, and it is related to growth and transfer of conceptual knowledge (Star & Rittle-Johnson, 2008). Recently, Newton et al. (2020) hypothesized a particular continuum for how students develop strategy flexibility: first, students are able to demonstrate a preference for (given) efficient strategies (e.g., when presented worked-out examples of a conventional and a non-conventional, more efficient, strategy, they are able to select the non-conventional strategy as 'better'). Next, they gain knowledge of multiple solution strategies (e.g., when asked to circle all possible steps that could be done next in solving an equation, they identify the conventional and non-conventional strategy). In the last phases of the developmental continuum, students are able to actually use the non-conventional, more efficient strategies, when asked to solve a set of problems.

Research has indicated several instructional factors that may promote mathematical flexibility, including stimulating children to invent, reflect, and discuss strategies or representations (Blöte et al., 2001), exposure to multiple strategies or representations (Rittle-Johnson et al., 2012), comparing worked-out examples (Rittle-Johnson & Star, 2007), and teachers asking open questions (Star et al., 2015). Mathematics textbooks have been found to differ in their learning opportunities for mathematical flexibility, and these differences were reflected in students' flexibility (Sievert et al., 2019). In this special issue, Schiller and colleagues (2022) addressed the relationship between the presentation of fractions equivalent to one in mathematics textbooks (e.g., 4/4) and students' flexible understanding of alternative forms of one.

Some studies have compared direct instruction of different strategies or representations to implicit instruction with prompts to discover strategies but without explicit instruction (Heinze et al., 2018; Star & Rittle-Johnson, 2008). Both approaches improved mathematical flexibility and can even be combined. Game-based learning approaches have been shown to foster adaptive number knowledge (Brezovszky et al., 2019), although these have been compared with business as usual control conditions and specific mechanisms for support remain unclear. In conclusion, although mathematical flexibility is notoriously hard to foster, educational practices can stimulate it. In this special issue, Scheibling-Sève and colleagues (2022) present an intervention study focusing on multiple categorizations and investigated the effect on students' propotional reasoning performance and strategy use.

Furthermore, we already discussed research suggesting that the socio-mathematical context affects the likelihood that students show mathematically flexible behavior. An educational implication is that classrooms probably differ in the extent to which mathematical flexibility is valued. Factors such as the extent to which accuracy is valued over speed, the extent to which mental computation is valued over written computation, or the extent to which finding mathematically elegant approaches or solutions are valued may be factors that teachers can consider in order to stimulate mathematical flexibility.

An important consideration is the interaction between individual factors and flexibility learning gains. One such factor is prior knowledge: as discussed earlier, research findings suggest that developing flexibility is very difficult for students with low prior knowledge. Indeed, students' prior knowledge has been found to be related to students' flexibility gains (Star et al., 2015), as was grade level (Brezovszky et al., 2019). However, more research is needed to determine when, how best, and to whom to provide instruction on mathematical flexiblity. Gender is another relevant factor. As said, if gender differences in mathematical flexibility are reported, boys show more mathematical flexibility than girls. However, the intervention study of Star and colleagues (2015) showed that girls had higher flexibility gains after instruction than boys. Clearly, this calls for further research.

The Current Special Issue

The proposed special issue provides a broad view of state-of-the-art research on mathematical flexibility. Mathematical flexibility is examined in different populations (including primary school students, secondary school students, and adults), in different domains (including whole-number arithmetic, fractions, proportional reasoning, and algebra), and with different methodologies (verbal reports, online tests, choice/no-choice design). Importantly, the papers also take on different theoretical perspectives. At the core of most of the studies involved in this special issue is the common conceptualization of mathematical flexibility as the ability to know and use different strategies and to select the most appropriate one for a given problem. However, this focus is broadened in several ways: by addressing reasoning about procedures, formulas, or magnitudes, by zooming in on the complex relationship between strategy use and accuracy, by attempts to measure whether children know more than they show, by focusing on multiple representations of a problem situation, by taking the affective side of mathematical flexibility into account, and finally by using a quite different conceptualization of mathematical flexibility: the flexible understanding of number. Furthermore, the studies shed light on the roles of domain-general and domain-specific skills play in mathematical flexibility as well as on the effects of an intervention. Collectively, the special issue aims to provide a broader understanding of the influences, subcomponents, and malleability of mathematical flexibility. We believe that this collection of empirical papers provides new insights and provokes new questions regarding the nature of mathematical flexibility. We hope it is of interest to researchers interested in mathematical cognition and learning and mathematics education alike.

Funding: The work was partly funded by a personal grant from the Dutch Research Council (NWO) for Hickendorff: 016.Veni.195.166/6812. These funding sources did not play a role in the content of the article.

Acknowledgments: The authors have no additional (i.e., non-financial) support to report.

Competing Interests: The authors have declared that no competing interests exist.

References

Acevedo Nistal, A., Van Dooren, W., Clarebout, G., Elen, J., & Verschaffel, L. (2009). Conceptualising, investigating and stimulating representational flexibility in mathematical problem solving and learning: A critical review. *ZDM – Mathematics Education, 41*(5), 627–636. https://doi.org/10.1007/s11858-009-0189-1

Baroody, A. J. (2003). The development of adaptive expertise and flexibility: The integration of conceptual and procedural knowledge. In A. J. Baroody & A. Dowker (Eds.), *The development of arithmetic concepts and skills: Constructing adaptive expertise* (pp. 1–33). Lawrence Erlbaum Associates.

Baroody, A. J., & Gannon, K. E. (1984). The development of the commutativity principle and economical addition strategies. *Cognition and Instruction, 1*(3), 321–339. https://doi.org/10.1207/s1532690xci0103_3

Baroody, A. J., Torbeyns, J., & Verschaffel, L. (2009). Young children's understanding and application of subtraction-related principles. *Mathematical Thinking and Learning, 11*(1–2), 2–9. https://doi.org/10.1080/10986060802583873

Blöte, A. W., van der Burg, E., & Klein, A. S. (2001). Students' flexibility in solving two-digit addition and subtraction problems: Instruction effects. *Journal of Educational Psychology, 93*(3), 627–638. https://doi.org/10.1037/0022-0663.93.3.627

Brezovszky, B., McMullen, J., Veermans, K., Hannula-Sormunen, M. M., Rodríguez-Aflecht, G., Pongsakdi, N., Laakkonen, E., & Lehtinen, E. (2019). Effects of a mathematics game-based learning environment on primary school students' adaptive number knowledge. *Computers & Education, 128*, 63–74. https://doi.org/10.1016/j.compedu.2018.09.011

Cragg, L., & Gilmore, C. (2014). Skills underlying mathematics: The role of executive function in the development of mathematics proficiency. *Trends in Neuroscience and Education, 3*(2), 63–68. https://doi.org/10.1016/j.tine.2013.12.001

Dowker, A., Sarkar, A., & Looi, C. Y. (2016). Mathematics anxiety: What have we learned in 60 years? *Frontiers in Psychology, 7*(APR), Article 508. https://doi.org/10.3389/fpsyg.2016.00508

Eaves, J., Attridge, N., & Gilmore, C. (2022). The role of domain-general and domain-specific skills in the identification of arithmetic strategies. *Journal of Numerical Cognition, 8*(3), 335–350. https://doi.org/10.5964/7459

Ellis, S. (1997). Strategy choice in sociocultural context. *Developmental Review, 17*(4), 490–524. https://doi.org/10.1006/drev.1997.0444

Fazio, L. K., DeWolf, M., & Siegler, R. S. (2016). Strategy use and strategy choice in fraction magnitude comparison. *Journal of Experimental Psychology: Learning, Memory, and Cognition, 42*(1), 1–16. https://doi.org/10.1037/xlm0000153

Garcia Coppersmith, J., & Star, J. R. (2022). A complicated relationship: Examining the relationship between flexible strategy use and accuracy. *Journal of Numerical Cognition, 8*(3), 382–397. https://doi.org/10.5964/jnc.7601

Halme, H., Trezise, K., Hannula-Sormunen, M. M., & McMullen, J. (2022). Characterizing mathematics anxiety and its relation to performance in routine and adaptive tasks. *Journal of Numerical Cognition, 8*(3), 414–429. https://doi.org/10.5964/jnc.7675

Heinze, A., Arend, J., Gruessing, M., & Lipowsky, F. (2018). Instructional approaches to foster third graders' adaptive use of strategies: An experimental study on the effects of two learning environments on multi-digit addition and subtraction. *Instructional Science, 46*(6), 869–891. https://doi.org/10.1007/s11251-018-9457-1

Heinze, A., Marschick, F., & Lipowsky, F. (2009). Addition and subtraction of three-digit numbers: Adaptive strategy use and the influence of instruction in German third grade. *ZDM – Mathematics Education, 41*(5), 591–604. https://doi.org/10.1007/s11858-009-0205-5

Heinze, A., Star, J. R., & Verschaffel, L. (2009). Flexible and adaptive use of strategies and representations in mathematics education. *ZDM – Mathematics Education, 41*(5), 535–540. https://doi.org/10.1007/s11858-009-0214-4

Hickendorff, M. (2018). Dutch sixth graders' use of shortcut strategies in solving multidigit arithmetic problems. *European Journal of Psychology of Education, 33*(4), 577–594. https://doi.org/10.1007/s10212-017-0357-6

Hickendorff, M. (2020). Fourth graders' adaptive strategy use in solving multidigit subtraction problems. *Learning and Instruction, 67*, Article 101311. https://doi.org/10.1016/j.learninstruc.2020.101311

Hickendorff, M. (2022). Flexibility and adaptivity in arithmetic strategy use: What children know and what they show. *Journal of Numerical Cognition, 8*(3), 367–381. https://doi.org/10.5964/jnc.7277

Hickendorff, M., Torbeyns, J., & Verschaffel, L. (2018). Grade-related differences in strategy use in multidigit division in two instructional settings. *British Journal of Developmental Psychology, 36*(2), 169–187. https://doi.org/10.1111/bjdp.12223

Hickendorff, M., van Putten, C. M., Verhelst, N. D., & Heiser, W. J. (2010). Individual differences in strategy use on division problems: Mental versus written computation. *Journal of Educational Psychology, 102*(2), 438–452. https://doi.org/10.1037/a0018177

Kieran, C. (1992). Learning and teaching of school algebra. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 390–419). Simon & Schuster.

Krutetskii, V. A. (1976). *The psychology of mathematical ability in schoolchildren.* University of Chicago Press.

Luwel, K., Beem, A. L., Onghena, P., & Verschaffel, L. (2001). Using segmented linear regression models with unknown change points to analyze strategy shifts in cognitive tasks. *Behavior Research Methods, Instruments, & Computers, 33*(4), 470–478. https://doi.org/10.3758/BF03195404

Luwel, K., Onghena, P., Torbeyns, J., Schillemans, V., & Verschaffel, L. (2009). Strengths and weaknesses of the choice/no-choice method in research on strategy use. *European Psychologist, 14*(4), 351–362. https://doi.org/10.1027/1016-9040.14.4.351

McMullen, J., Brezovszky, B., Hannula-Sormunen, M. M., Veermans, K., Rodríguez-Aflecht, G., Pongsakdi, N., & Lehtinen, E. (2017). Adaptive number knowledge and its relation to arithmetic and pre-algebra knowledge. *Learning and Instruction, 49*, 178–187. https://doi.org/10.1016/j.learninstruc.2017.02.001

Newton, K. J., Lange, K., & Booth, J. L. (2020). Mathematical flexibility: Aspects of a continuum and the role of prior knowledge. *Journal of Experimental Education, 88*(4), 503–515. https://doi.org/10.1080/00220973.2019.1586629

Nunes Carraher, T., Carraher, D. W., & Schliemann, A. D. (1985). Mathematics in the streets and in schools. *British Journal of Developmental Psychology, 3*(1), 21–29. https://doi.org/10.1111/j.2044-835X.1985.tb00951.x

Obersteiner, A., Alibali, M. W., & Marupudi, V. (2022). Comparing fraction magnitudes: Adults' verbal reports reveal strategy flexibility and adaptivity, but also bias. *Journal of Numerical Cognition, 8*(3), 398–413. https://doi.org/10.5964/jnc.7577

Peters, G., De Smedt, B., Torbeyns, J., Ghesquière, P., & Verschaffel, L. (2010). Adults' use of subtraction by addition. *Acta Psychologica, 135*(3), 323–329. https://doi.org/10.1016/j.actpsy.2010.08.007

Peters, G., De Smedt, B., Torbeyns, J., Ghesquière, P., & Verschaffel, L. (2012). Children's use of subtraction by addition on large single-digit subtractions. *Educational Studies in Mathematics, 79*(3), 335–349. https://doi.org/10.1007/s10649-011-9308-3

Rasmussen, C., Ho, E., & Bisanz, J. (2003). Use of the mathematical principle of inversion in young children. *Journal of Experimental Child Psychology, 85*(2), 89–102. https://doi.org/10.1016/S0022-0965(03)00031-6

Rittle-Johnson, B., Siegler, R. S., & Alibali, M. W. (2001). Developing conceptual understanding and procedural skill in mathematics: An iterative process. *Journal of Educational Psychology, 93*(2), 346–362. https://doi.org/10.1037/0022-0663.93.2.346

Rittle-Johnson, B., & Star, J. R. (2007). Does comparing solution methods facilitate conceptual and procedural knowledge? An experimental study on learning to solve equations. *Journal of Educational Psychology, 99*(3), 561–574. https://doi.org/10.1037/0022-0663.99.3.561

Rittle-Johnson, B., Star, J. R., & Durkin, K. (2012). Developing procedural flexibility: Are novices prepared to learn from comparing procedures? *British Journal of Educational Psychology, 82*(3), 436–455. https://doi.org/10.1111/j.2044-8279.2011.02037.x

Scheibling-Sève, C., Gvozdic, K., Pasquinelli, E., & Sander, E. (2022). Enhancing cognitive flexibility through a training based on multiple categorization: Developing proportional reasoning in primary school. *Journal of Numerical Cognition, 8*(3), 443–472. https://doi.org/10.5964/jnc.7661

Schiller, L. K., Fan, A., & Siegler, R. S. (2022). The power of one: The importance of flexible understanding of an identity element. *Journal of Numerical Cognition, 8*(3), 430–442. https://doi.org/10.5964/jnc.7593

Schneider, M., Rittle-Johnson, B., & Star, J. R. (2011). Relations among conceptual knowledge, procedural knowledge, and procedural flexibility in two samples differing in prior knowledge. *Developmental Psychology, 47*(6), 1525–1538. https://doi.org/10.1037/a0024997

Siegler, R. S., & Lemaire, P. (1997). Older and younger adults' strategy choices in multiplication: Testing predictions of ASCM using the choice/no-choice method. *Journal of Experimental Psychology: General, 126*(1), 71–92. https://doi.org/10.1037/0096-3445.126.1.71

Sievert, H., van den Ham, A.-K., Niedermeyer, I., & Heinze, A. (2019). Effects of mathematics textbooks on the development of primary school children's adaptive expertise in arithmetic. *Learning and Individual Differences, 74*, Article 101716. https://doi.org/10.1016/j.lindif.2019.02.006

Star, J. R., Newton, K., Pollack, C., Kokka, K., Rittle-Johnson, B., & Durkin, K. (2015). Student, teacher, and instructional characteristics related to students' gains in flexibility. *Contemporary Educational Psychology, 41*, 198–208. https://doi.org/10.1016/j.cedpsych.2015.03.001

Star, J. R., & Rittle-Johnson, B. (2008). Flexibility in problem solving: The case of equation solving. *Learning and Instruction, 18*(6), 565–579. https://doi.org/10.1016/j.learninstruc.2007.09.018

Star, J. R., & Seifert, C. (2006). The development of flexibility in equation solving. *Contemporary Educational Psychology, 31*(3), 280–300. https://doi.org/10.1016/j.cedpsych.2005.08.001

Threlfall, J. (2009). Strategies and flexibility in mental calculation. *ZDM – Mathematics Education, 41*(5), 541–555. https://doi.org/10.1007/s11858-009-0195-3

Timmermans, R. E., Van Lieshout, E. C. D. M., & Verhoeven, L. (2007). Gender-related effects of contemporary math instruction for low performers on problem-solving behavior. *Learning and Instruction, 17*(1), 42–54. https://doi.org/10.1016/j.learninstruc.2006.11.005

Torbeyns, J., De Smedt, B., Ghesquière, P., & Verschaffel, L. (2009). Acquisition and use of shortcut strategies by traditionally schooled children. *Educational Studies in Mathematics, 71*(1), 1–17. https://doi.org/10.1007/s10649-008-9155-z

Torbeyns, J., Hickendorff, M., & Verschaffel, L. (2015). Identificatie van strategiegebruik bij rekenen via niet-verbale methoden: Mogelijkheden, beperkingen en inhoudelijke bevindingen. Een inleiding. *Pedagogische Studiën, 92*(1), 2–8.

Torbeyns, J., Hickendorff, M., & Verschaffel, L. (2017). The use of number-based versus digit-based strategies on multi-digit subtractions: 9-12-year-olds' strategy use profiles and task performance. *Learning and Individual Differences, 58*, 64–74. https://doi.org/10.1016/j.lindif.2017.07.004

Torbeyns, J., Verschaffel, L., & Ghesquière, P. (2006). The development of children's adaptive expertise in the number domain 20 to 100. *Cognition and Instruction, 24*(4), 439–465. https://doi.org/10.1207/s1532690xci2404_2

Van Der Auwera, S., De Smedt, B., Torbeyns, J., & Verschaffel, L. (2022). Adults' use of subtraction by addition and its association with executive functions. *Journal of Numerical Cognition, 8*(3), 351–366. https://doi.org/10.5964/jnc.7271

Van Der Auwera, S., Torbeyns, J., De Smedt, B., Verguts, G., & Verschaffel, L. (2022). The remarkably frequent, efficient, and adaptive use of the subtraction by addition strategy: A choice/no-choice study in fourth- to sixth-graders with varying mathematical achievement levels. *Learning and Individual Differences, 93*, Article 102107. https://doi.org/10.1016/j.lindif.2021.102107

Verschaffel, L., Luwel, K., Torbeyns, J., & Van Dooren, W. (2009). Conceptualizing, investigating, and enhancing adaptive expertise in elementary mathematics education. *European Journal of Psychology of Education, 24*(3), 335–359. https://doi.org/10.1007/BF03174765

Verschaffel, L., Luwel, K., Torbeyns, J., & Van Dooren, W. (2011). Analyzing and developing strategy flexibility in mathematics education. In J. Elen, E. Stahl, R. Bromme, & G. Clarebout (Eds.), *Links between beliefs and cognitive flexibility* (pp. 175-197). Springer, Dordrecht.

Xu, L., Liu, R.-D., Star, J. R., Wang, J., Liu, Y., & Zhen, R. (2017). Measures of potential flexibility and practical flexibility in equation solving. *Frontiers in Psychology, 8*(AUG), Article 1368. https://doi.org/10.3389/fpsyg.2017.01368

Journal of Numerical Cognition (JNC) is an official journal of the Mathematical Cognition and Learning Society (MCLS).

leibniz-psychology.org

PsychOpen GOLD is a publishing service by Leibniz Institute for Psychology (ZPID), Germany.

PsychOpen GOLD