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Visual and Symbolic Representations as Components of Algebraic Reasoning

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Supplementary Materials: Data, Materials [see Index of Supplementary Materials]

Abstract

Sixty (35 girls) ninth graders were assessed on measures of algebraic reasoning and usage of visual and symbolic representations (with a prompt for visual use) to solve equations and inequalities. The study grouped visual representations into two categories: arithmetic-visual, which entailed the use of real-world objects to represent specific values of variables, and algebraic-visual, which involved formal representations like the number line and the coordinate plane. Symbolic representations, on the other hand, encompassed the use of standard algorithms to solve equations, such as changing the place of terms in an equation. The results reveal that the use of algebraic visuals, as opposed to arithmetic visuals, was associated with enhanced algebraic reasoning. Further, although the students initially relied on standard algorithms to explain equations and inequalities, they could produce accurate algebraic-visual representations when prompted. These findings suggest that students have multiple representations of equations and inequalities but only express visual representations when asked to do so. In keeping with the general relationship between visuospatial abilities and mathematics, self-generated algebraic-visual representations partially mediated the relation between overall mathematics achievement and algebraic reasoning.

Keywords

mathematics achievement, arithmetic, algebra, visual representation, symbolic representation

Competence with algebra is the gateway to advanced mathematics and provides the foundation for work in science, technology, engineering, and mathematics (STEM) disciplines and many other occupations (Adelman, 2006; Bynner, 1997; National Mathematics Advisory Panel, 2008; RAND Mathematics Study Panel, 2003). However, many students struggle with the transition from an arithmetical mindset, or thinking in terms of specific cases, to an algebraic perspective or using generalized models to reason algebraically (Ely & Adams, 2012; Kaput et al., 2017; Kilpatrick & Izsak, 2008; Stacey & MacGregor, 1999; Stein et al., 2011). The latter refers to the ability to express general patterns and relations among variables systematically using mathematical symbols (Blanton & Kaput, 2005; Kaput, 2017). Many intervention studies and meta-analyses have focused on instructional strategies to improve struggling students' algebraic reasoning (e.g., Bone et al., 2021; Bouck et al., 2019; Foegen, 2008; Haas, 2005; Hughes et al., 2014). These studies are essential because they provide evidence-based approaches to support students in making the transition to algebraic thinking, which is critical for success in both academic and professional contexts.

One of the recurring themes in intervention studies is the use of manipulatives and object pictures to represent quantity and variables, followed by a gradual transition to standard algebraic form (e.g., Long et al., 2021; Satsangi et al., 2016, 2018). However, these studies often overlook assessments of students' preexisting mental models of algebraic problems. Although algebra tiles or diagrams are occasionally employed to teach algebraic expressions in such studies (e.g., Alamian et al., 2020; Castro, 2017; Nagashima et al., 2021), pretests are rarely conducted to determine the nature of students' existing representations of these expressions, such as visual representations, and how these representations change over the course of the intervention.

That is a substantial limitation because the use of student-generated representations in mathematics is crucial for tracking students' progress and supporting their mathematical development (Selling, 2016), in contrast to using researcher-provided representations (Guo et al., 2020). For instance, Terwel et al. (2009) showed that students who learned to generate their own visuals to solve mathematics problems outperformed those who learned how to use researcher-provided visuals. In the realm of algebra, the ability to use visuospatial abilities to represent algebraic relationships may provide insights into the sophistication of students' algebraic thinking, as opposed to simple reinventing algebraic representations (e.g., plotting functions in coordinate space).

Consequently, we studied the relationship between students' algebraic reasoning and their self-generated visual and symbolic representations of equations and inequalities following their initial problem-solving attempts. We focused on student-generated visual representations because they should reveal the mental models students used to understand algebraic relationships, which may be an integral part of their algebraic reasoning. Further, this approach would provide insight into how students utilize their visuospatial abilities to understand mathematics, which is particularly relevant given the well-documented relationship between visuospatial abilities and various mathematics outcomes (Atit et al., 2022; Casey & Ganley, 2021; Geary et al., 2023; Hawes & Ansari, 2020; Mix, 2019).

Although psychometric visuospatial measures are useful, they do not provide a comprehensive understanding of how students employ their visuospatial abilities to comprehend mathematics. Therefore, studying student-generated visual representations of algebraic relationships offers a unique perspective on the role of visuospatial abilities in mathematical development.

Different Types of Mathematical Representations

Mathematical ideas can be communicated and demonstrated in various ways using representations (Greeno & Hall, 1997; Pape & Tchoshanov, 2001). The ability to produce a representation, such as a visual diagram of a linear relation between x and y, has been shown to reflect students' levels of understanding of the associated concept (Greeno & Hall, 1997). Indeed, many studies have established the association between students' use of visual and symbolic representations and their performance in specific mathematical domains (e.g., Boonen et al., 2014; Hegarty & Kozhevnikov, 1999; van Garderen & Montague, 2003).

Current mathematical knowledge is associated with variations in the representation and problem-solving methods used across different achievement levels (Deliyianni et al., 2016; Kaizer & Shore, 1995; Newton et al., 2020; van Garderen et al., 2013). Lower-achieving students tend to show less variation in their representation and problem-solving methods across different types of problems (e.g., Krawec, 2014; Siegler, 1998). For instance, Heirdsfield and Cooper (2002) found that lower-achieving students were less likely to use alternative representations while doing computations (e.g., 28 + 35: 8 + 5 = 13, 20 + 30 = 50, 63 or 20 + 30 = 50, 8 + 5 = 13, 63). For these types of computations, decomposing the problem into subsets facilitates problem-solving and provides an advantage in arithmetic learning (Geary, 2011; Geary et al., 2007). However, the variation in how students flexibly represent algebraic problems, particularly algebraic equations and inequalities, is not well understood.

Visual Representations

Visual representations refer to those involving the use of imagery to illustrate problems and find solutions (Arcavi, 2003; Presmeg, 1986), and their use may contribute to the well-documented relation between visuospatial abilities and mathematics outcomes (Casey & Ganley, 2021; Geary et al., 2021; Halpern et al., 2007; Mix, 2019). However, some nuances should be considered while examining this potential relation. First, as mentioned above, there is variation in the method of visual use, whether researcher-provided or self-generated (Guo et al., 2020), which can be illustrated through Bloom's Taxonomy.

Bloom's Taxonomy outlines six stages of learning outcomes: remembering, understanding, applying, analyzing, evaluating, and creating (Anderson et al., 2001). The construction of visual representations occurs in the application phase after students have a solid understanding of the underlying mathematics concepts. This understanding enables students to integrate and transfer these concepts into a visual form (Van Meter & Garner, 2005). In other words, self-generated visuals require a strong foundation in the associated mathematics domain and exposure to conventional researcher-provided visual models. The details of the self-generated visuals should reflect how the student understands the presented problem.

In addition to aiding comprehension, self-generated drawings can also contribute to mathematical development by serving as both an analysis and a solution medium. They make problems more concrete, allow for the rearrangement of the given information, abstract problem features, direct attention to problem semantics, and facilitate the correct representation of quantitative relationships (van Essen & Hamaker, 1990). Moreover, managing cognitive load during instruction is crucial due to limitations in working memory capacity, as posited by cognitive load theory (Chandler & Sweller, 1991), a framework focusing on human information processing and instructional materials. From this perspective, self-generated drawings of a problem could reduce working memory demands and thereby increase problem-solving efficiency (van Essen & Hamaker, 1990). Therefore, self-generated visual representations might be a valuable tool for enhancing mathematical comprehension and proficiency.

Third, the contents of self-generated visual representations used during problem-solving can provide valuable insights into the problem-solving process, including potential misunderstanding of mathematical content (Krawec, 2014; Moore & Carlson, 2012; Nagashima et al., 2021; van Garderen et al., 2013; van Garderen & Montague, 2003). For instance, van Garderen and Montague (2003) found that schematic representations, as opposed to pictorial ones, had a positive association with word problem performance, and the usefulness of the visual representations varied across different levels of mathematics achievement. Students with lower mathematics achievement often produced less systematic diagrams that did not accurately represent the core relations in the problems, unlike their high-achieving peers (van Garderen et al., 2013).

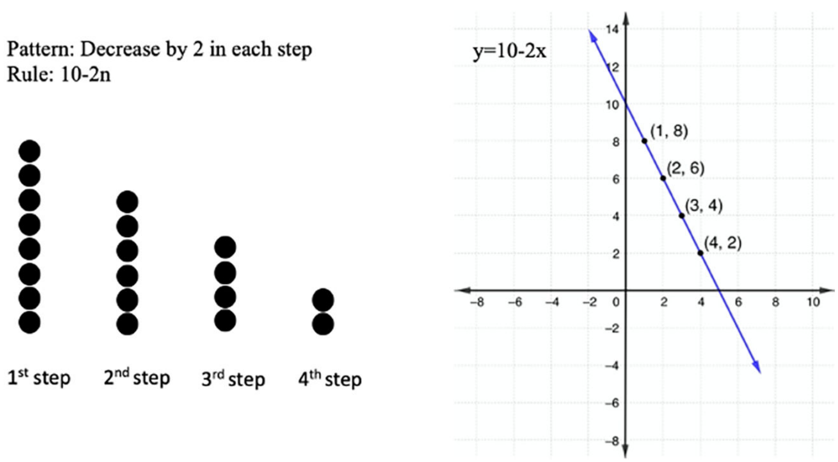
Despite these findings, the specific ways in which visuospatial abilities contribute to mathematics learning, particularly in algebra (Muchoko et al., 2019), are not fully understood (Barnes et al., 2014). To the best of our knowledge, no study has examined students' self-generated visual representations of algebraic problems. Some studies have examined researcher-provided drawings of algebraic concepts, but their impact on instructional outcomes is mixed. For instance, Ross and Willson (2012) found that teachers' use of alternative representations (e.g., written numerals or symbols) during algebra instruction contributed to students' learning, whereas the incorporation of visual representations such as graphs did not improve student outcomes. In contrast, Alamian et al. (2020) demonstrated that using visuals (e.g., diagrams) significantly reduced algebraic misconceptions. For instance, visual aids like diagrams and drawings significantly decreased the frequency of inappropriate interpretations of equations [e.g., given $(x - 4)(x + 1) = 5$, making both parenthetical values equal to 5] after an intervention embedding visuals into algebra instruction.

Mixed results may stem from individual differences in students' understanding of visuospatial representations of mathematical concepts and relationships. However, prior studies did not assess students' mental models of algebraic concepts before instruction, particularly through students' self-generated drawings, making it difficult to infer how students' models (including visual representations) changed as they gained competence in algebra. There is reason to believe that these representations do change with instruction. For instance, Selling's (2016) study of students' work on patterning problems, a potential precursor of algebraic thinking, indicates that students' mental representations become more sophisticated (e.g., simultaneous use of different visual representations, such as tables and drawings, showing the growth pattern) as they become more competent in solving these problems.

Arithmetic- and Algebraic-Visual Representations — Based on the types of representations used to teach and solve word and patterning problems (Tezcan, 2016), we anticipated that students would generate a combination of arithmetic (i.e., object-based visuals, including real-life objects used to substitute for specific values of variables) and algebraic (i.e., efficient formal visuals like the number line, algebraic tape diagrams or tiles, and coordinate planes) representations to solve algebra problems. The arithmetic-visual representations are used in teaching basic mathematics concepts, particularly patterning. For instance, students first learn patterns in arithmetic series using concrete materials (e.g., decrease by two in each step, for the visual, see Figure 1) in the fifth grade, followed by a gradual transition to symbolic notations used to present the rule of the arithmetic series in the seventh grade (e.g., 10-2n, for the visual, see Figure 1). Then, the algebraic instruction in the following years is based, in part, on earlier knowledge of arithmetic series (Akkan & Çakıroğlu, 2012).

Figure 1

Examples of Visual Representations



Note. Visuals of arithmetic series (left) and linear equations (right) for 8th-grade students. As early as 5th grade, students are expected to discern the pattern, which is a gradual decrease by 2, and by 7th grade, they should be able to write the rule, which is represented as 10-2n. By 8th grade, students are expected to apply this knowledge to extend it to linear relations. These examples were adapted from the Turkish Ministry of National Education's mathematics textbooks and were constructed using GeoGebra (Hohenwarter, 2002).

It is crucial to examine the use of algebraic-visual representations as they might aid in bridging the gap between arithmetic and algebraic reasoning, which is difficult to close (Herscovics & Linchevski, 1994). Many students continue to use an arithmetical perspective when attempting to solve algebra problems; for instance, some use substitution (e.g., substituting 6 for x to find the solution set of $7x < 42$) to solve such problems. Generating algebraic-visual representations can help differentiate between the two approaches because algebraic visuals can show generalized relations that cannot be represented arithmetically.

For instance, the visual representation used to teach the arithmetic series in Figure 1 cannot represent abstract numbers that do not have an object-based correspondence, such as irrational numbers (Patel & Varma, 2018), for the variable. In contrast, the coordinate plane in Figure 1 can. Further, arithmetic-visual representations cannot easily represent continuous relations: Each segment of the visual containing balls in Figure 1 describes a case for a specific value substituted for the variable (e.g., rule = 10 − 2n, n = 1, for the first part containing eight balls), emphasizing case-specificity. In contrast, the line on the coordinate plane continuously includes all possible values with one visual and better depicts the generalized relation. Here, it should be noted that using arithmetic series to support teaching algebra varies across countries. For instance, Turkish and Singapore standards include it, whereas the US standards do not (Akkan & Çakıroğlu, 2012).

In any case, it should be noted that the difference between arithmetic- and algebraic-visual representations might be analogous to students' word problem drawings of pictorial versus schematic representations mentioned earlier. While the pictorial representations depict an image of the story problems, schematic visuals show patterns within the problem, and only the latter has a significant positive relation with word problem performance (van Garderen et al., 2013).

Symbolic Representations

Mathematical concepts can also be represented symbolically (Kaput, 1987; Mainali, 2021), that is, using and manipulating mathematical symbols, including numbers, operations, connection signs, and algebraic signs, which will be referred to as symbolic representations (Anwar & Rahmawati, 2017; Kaput et al., 2017). Unlike visual representations, symbolic representations do not have corresponding visuospatial depictions. While working on algebraic representations, the two representation types are distinguished (Anwar & Rahmawati, 2017), and thus symbolic representations will be presented separately in the current study.

Working with algebraic equations and inequalities depends partly on competence in using symbolic representations arithmetically, such as using the distributive law (Maffia & Mariotti, 2020). Schneider et al. (2011) proposed that students who were good at symbolic manipulations [e.g., while solving the equation $3(y + 1) = 4(y + 1) + 2(y + 1)$, generating two different solution methods via symbolic manipulations] seemed to have a more solid understanding of algebraic equations. They could direct their attention to the most significant features of the equations and choose the most appropriate solution method more easily. However, their findings did not include inequalities or equations with two unknowns; thus, understanding the relation between symbolic representation and algebraic reasoning that includes inequalities and other types of equations requires further study.

Further, Ross and Willson (2012) found that teachers' use of symbolic representations during in-class activities was positively associated with algebraic understanding. That might be because the symbolic representation is a device to express the algebra concepts meaningfully (Ross & Willson, 2012). It should be noted, however, that they did not investigate the link between students' existing symbolic representations and their algebraic reasoning but rather the effect of using symbolic representations for in-class activities focused on the concepts of *variable* and *change*.

Here, we assume that students can represent and solve problems in multiple ways, including generating symbolic and visual representations. This assumption follows Siegler's (1996, 1998) overlapping waves and strategy choice model, whereby different problem-solving strategies are used flexibly based on problem requirements. In other words, even if students solve algebra problems using the conventional symbolic methods, this does not preclude the existence of alternative symbolic and visual approaches that might only be expressed under some conditions, including when being prompted to use them.

Current Study

In the current study, we aimed to investigate how students' self-generated visual and symbolic representations were related to their algebraic reasoning, particularly following prompts to use visualizations after solving algebraic inequality and equation problems. While we examined the relationship between all types of representations and algebraic reasoning, our main focus was on the potential predictive power of self-generated visual representations. We hypothesized that visual representations, specifically those related to algebra, would be better predictors of algebraic reasoning than arithmetic visuals, as has been found in previous research on word problems (Krawec, 2014; Moore & Carlson, 2012; Nagashima et al., 2021; van Garderen et al., 2013; van Garderen & Montague, 2003). To control for the influence of general ability and mathematics achievement, which are known to impact mathematics development (Deary et al., 2007; Geary et al., 2017), we included these factors as covariates in our analyses.

Our focus on visual representations assumed that the ability to form sophisticated visuospatial representations of mathematical relations mediates mathematics gains (Gilligan et al., 2017; Möhring et al., 2021; Young et al., 2018). On this view, the relationship between overall mathematics achievement and algebraic reasoning is partially or fully mediated by the quality of students' self-generated visual representations. Furthermore, we proposed a bidirectional relationship between algebraic reasoning and self-generated algebraic-visual representations, where each factor could influence the other. Therefore, we conducted analyses to test the hypothesis that self-generated algebraic visualizations mediate the relationship between overall mathematics achievement and algebraic reasoning, as well as the alternative hypothesis that algebraic reasoning mediates the relationship between mathematics achievement and self-generated algebraic visualizations. Due to the correlational nature of our data, we could not establish causality, and both hypotheses were considered equally plausible.

Method

Participants

The participants were a convenience sample of 60 ninth-grade students in Turkey (*N* girls: 35, *N* boys: 25). They were recruited through a variety of channels, including announcements made by high school principals and mathematics teachers, posts on social media, and bulletin boards on a popular website that displayed online/virtual experiments (http://bilimsevercocuklar.com/). The participants were all native Turkish speakers, aged 15 years (M = 15.16, SD = 0.54, range 14.17 to 16.58 yrs.), hailing from 17 cities and 40 high schools. The demographic information was gathered through online parental surveys, revealing diverse educational backgrounds, as presented in Table 1.

Table 1

Distribution of Parental Education

	Primary	Middle	High	College	Graduate[a]
Father	22% (13)	8% (5)	23% (14)	37% (22)	10% (6)
Mother	25% (15)	12% (7)	33% (20)	23% (14)	7% (4)

Note. Primary = 5 years of schooling; middle = 8 years of schooling; high = 12 years of schooling; College = 13-18 years of schooling.
[a]graduate = at least one graduate school.

Measures

General Ability

We used the Pearson online version of Raven's Progressive Matrices (Raven's 2) as an index of general ability (Pearson Clinical Assessment, n.d.). Based on the original (Raven & Court, 1998), the test was designed to evaluate students' ability to recognize patterns in a sequence of visual patterns with a missing piece and select the correct option from five choices to complete the sequence. The test was self-phased with a 20-minute maximum duration and provided standardized scores (M = 100, SD = 15). One student did not take the test. To address this missing data, we employed a model-based imputation technique using scores from algebraic reasoning, mathematics achievement, and representation as predictors. This method ensured that the missing score was accurately imputed based on relevant factors.

Mathematics Achievement

High School Entrance Exam (LGS, Turkish acronym), a rigorous and high-stakes test in Turkey, was taken by 1,472,088 students in 2020, of which only 212,485 were granted admission to high school based on their scores. This comprehensive exam assesses students' knowledge in mathematics, science, Turkish, revolution history and Kemalism, religious culture and moral knowledge, and foreign language. The Turkish Ministry of National Education (2020) published a report detailing the exam results (for descriptive statistics, see Table 5, p. 19), revealing that the mean score for all test-takers was M = 286.35 out of 500 (SD = 69.22). For the current study, we used the 20-item mathematics subsection of LGS as an index of mathematics achievement (for all test-takers, M = 9.96, SD = 4.42). The mean mathematics score of the participants (for the current sample: M = 10.35, SD = 5.52) was moderately higher (d = 0.20) than that of all test takers.

The mathematics subsection of the LGS included questions covering algebra, data and probability, and radical numbers. Three questions, which constituted fifteen percent of the exam, involved algebra. However, it is worth noting that the algebraic questions in the exam differed from those in the algebraic interview, as they were multiple-choice and involved measurement concepts such as determining the area, length, and circumference of shapes, whereas the interview focused purely on algebraic concepts.

Algebra Interview

During the interview, the students were asked eleven questions related to their comprehension of variables, and there was no time limit. Each question was designed to explore a different aspect of students' understanding of variables.

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However, we paid extra attention to three questions for two reasons. Firstly, these questions involved variables that could have multiple values, such as x = 2 or x = 0 for the equation 5x < 30. Secondly, the questions could be solved using either visual aids like a number line or symbolic representations such as manipulating symbols in equations:

> Q1. What values of x make the following statement true: $5x$ is less than 30?
>
> Q2. What happens to the value of y as x increases in value given the equation $y = 2x + 3$?
>
> Q3. Ben said, "$w + 3$" is less than "$5 + w$". Circle one: (a) Always true, (b) Sometimes true, (c) Never true

Once the students had provided their initial response, typically involving standard symbolic algorithms, we inquired whether they could represent the equations or inequalities in an alternative way, explicitly prompting the use of visual representations. In cases where the students did not have a visual representation, we emphasized that such representations were not compulsory and that other forms of representation were acceptable. Subsequently, some students responded with additional symbolic representations, while others employed various forms of visual representation.

Coding Algebraic Reasoning — We utilized a three-category rubric, comprising 0 points, 1 point, and 2 points, to evaluate the initial responses to the three questions. The categories are explained in detail in Table A1 in the Appendix. A score of 0 points indicates the use of an erroneous method (such as incorrectly interchanging the coefficient and variable) or failure to provide a response. A score of 1 point signifies that the student provided answers that were partly accurate, with the solution method containing missing components or not being algebraic (such as substitution). A score of 2 points indicates that the student provided entirely correct answers, with the techniques employed being advanced, including generalizations. To compute the overall algebraic reasoning score, we summed up the scores across the three items, resulting in an average of 3.17 (SD = 1.70; range = 0 to 6).

Coding Algebraic Representations — We used two dimensions to evaluate the representations provided by students in response to the follow-up question. The first dimension focused on the type of representation used, which fell into three categories: algebraic-visual, arithmetic-visual, and symbolic. Algebraic-visual included solutions that used algebraic representations, including the number line, coordinate plane, and algebraic diagrams and tiles. Arithmetic-visual involved using real-life examples such as apples and balls to represent specific values for variables. Symbolic representation involved using and manipulating symbols in algebraic expressions.

The second dimension we considered was the accuracy of the response, which was scored as 0 (no targeted representation), 1 (partially accurate representation), or 2 (accurate and correct representation). The maximum score for each representation category was 6, but the scores were mutually exclusive. This means that if a student correctly used algebraic-visual representations for all three questions, their score would be 6 for algebraic-visual and 0 for arithmetic-visual and symbolic. Examples of student responses are presented in Table 2.

Procedure

The study was conducted online via Zoom due to COVID-19, with the legal guardians' permission. The session began with an algebra interview, where the questions were presented on the screen, and the students solved them on blank papers. They showed their work on the screen and explained their answers. Once the interview was completed, the students took pictures of their work and sent them to the researchers via phone messages. Following the interview, the students completed Raven's test digitally, with the researchers providing remote control. This study was approved by the local institution's ethics committee (IRB# E-84391427-050.01.04-18335/ 2021-31).

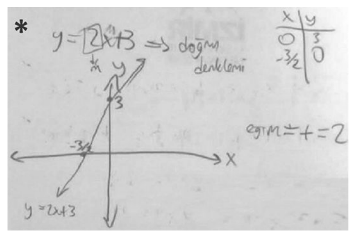
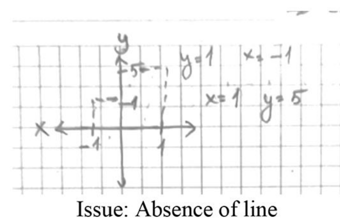
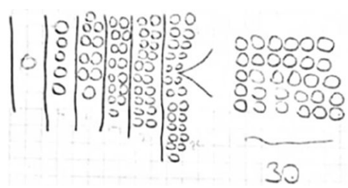
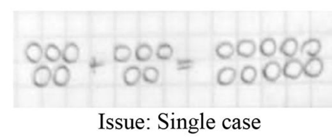
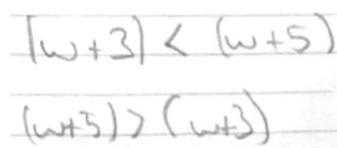
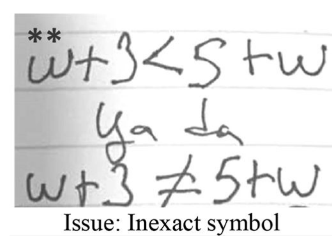
Inter-Rater Reliability

Three graduate students, who had at least a bachelor's degree in mathematics education and experience teaching diverse students, coded the students' responses. The first author, who was also one of the coders, developed the rubrics based on research findings. According to the literature, students use two solution methods when solving algebraic problems:

algebraic and non-algebraic (Stacey & MacGregor, 1999). Algebraic solutions involve generalized methods of algebraic thinking, while non-algebraic solutions involve arithmetical methods such as trial and error and substitution (Ely & Adams, 2012; Kilpatrick & Izsak, 2008; Stacey & MacGregor, 1999). The rubric for algebraic reasoning coding was developed based on this difference in solution methods, and full credit was given only if the solution methods were accurate and followed the algebraic route.

Table 2

Examples of Different Representation Types and Scoring

		Scoring		
Category	**Question**	**2 Points**	**1 Point**	**0 Points**
Algebraic	What happens to the value of y as x increases in value give the equation y = 2*x* + 3?			NR
Arithmetic	What values of x make the following statement true: 5*x* is less than 30?			NR
Symbolic	Ben said, "*w* + 3" is less than "5 + *w*." Circle one: *Always true *Sometimes true *Never true			NR

Note. Examples were students' responses. Algebraic = algebraic visual representation; Arithmetic = arithmetic visual representation; Symbolic = symbolic representation. *Doğru denklemi = line equation; eğim = slope. **ya da = or. NR = no response of the targeted representation.

The development of the algebraic representation codebook was based on the primary and middle school curriculum and standards (Ministry of National Education, 2022) and books published by the Turkish Ministry of Education. Arithmetic representations were used to teach arithmetic series in mathematics textbooks, while standard algebraic representations included algebraic diagrams and tiles, the number line, and the coordinate plane (Böge & Akıllı, 2021; Çağlayan et al., 2021; Cırıtcı et al., 2019; Maviş et al., 2021; Oğan & Öztürk, 2021).

The coding was also informed by previous research on the use of two types of representations in word problems: schematic and pictorial (Krawec, 2014; van Garderen, 2006; van Garderen & Montague, 2003; van Garderen et al., 2013). Schematic representations capture the full patterns of relations in word problems, while pictorial representations do not. This distinction is analogous to the use of algebraic versus arithmetical representations, as algebraic-visual representations are conventionally used in algebra textbooks and capture the entire patterns of relations, while arithmetical representations can only capture specific aspects of the patterns. Symbolic representations were based on articles focusing on symbolic representations of algebraic equations (Anwar & Rahmawati, 2017; Senk & Thompson, 2006).

These representations require the manipulation of numbers, operations, and equations in problems and have been shown to be highly related to algebra. (Newton et al., 2020).

To ensure the reliability of the coding process, two researchers independently coded all responses using the rubric. Any discrepancies were discussed between the two coders, and if they could not agree on a code, a third researcher was consulted. The final score was determined by the third researcher if their code matched one of the initial coder's codes. There were no instances where all three coders disagreed. The initial coders had a high level of agreement, with 92% to 95% agreement on the algebraic reasoning measure and 75% to 85% agreement on the algebraic representations scores. Any disagreements were resolved using the aforementioned method.

Data Analytical Plan

The study aimed to test the hypothesis that algebraic reasoning is related to self-generated visual representations of the associated concept, and this relation is not due to general cognitive ability or general mathematics achievement. To test this hypothesis, algebraic reasoning was regressed on general mathematics achievement, general ability scores, and the three representation types (algebraic-visual, arithmetic-visual, symbolic).

We performed post hoc power analysis to ensure that the sample size was sufficient using the GPower software package (Faul & Erdfelder, 1992). The statistical test was linear multiple regression: fixed model, R^2 deviation from zero with input parameters of total sample size, number of predictors, effect size f^2, and α err prob. The sample size was 60, and the number of predictors was 5. We entered the squared multiple correlation value taken from the regression analysis (i.e., R^2) in GPower to obtain the effect size (f^2). The recommended thresholds for interpretation were small (f^2 = .02), medium (f^2 = .15), and large (f^2 = .35) (Cohen, 1977). Lastly, the alpha level was .05.

Two mediation analyses were used in the study. The first analysis aimed to determine if generating algebraic-visual representation mediated the link between general mathematics achievement and algebraic reasoning, which would stem from the relation between visuospatial abilities and mathematics. The second analysis aimed to test if algebraic reasoning mediated the association between overall mathematics achievement and generating algebraic-visual representations. The analyses were conducted using R (R Core Team, 2022), and mediation analyses were done using the *lavaan* package (Rosseel, 2012). Data and codes for the study can be found online (see Supplementary Materials).

Results

Algebraic Reasoning and Visual Representations

Table 3 presents the correlations among variables in the study. The results reveal that higher algebraic reasoning scores were significantly associated with accurate algebraic visualizations (r = .58, p < .01) but showed a non-significant correlation with arithmetic-visual representation and symbolic representation. Further, generating algebraic-visual representations was negatively correlated with symbolic representation (r = -.25, p < .01), which likely reflects the mutually exclusive coding of the use of one strategy or another.

Table 3

Correlations Among Variables

Variable	*M*	*SD*	1	2	3	4	5
1. General ability	109.79	14.07					
2. Mathematics Achievement	10.35	5.52	0.73**				
3. Algebraic Reasoning	3.17	1.70	0.59**	.074**			
4. Algebraic-visual Representation	0.98	1.63	0.38**	0.56**	0.58**		
5. Arithmetic-visual Representation	0.73	1.42	0.23	0.07	0.04	-0.21	
6. Symbolic Representation	0.85	1.27	0.21	0.21	0.09	-.25*	-0.10

*p < .05. **p < .01.

Table 4 shows the results of a multiple regression analysis predicting algebraic reasoning from general ability, mathematics achievement, and three types of representations (algebraic-visual, arithmetic-visual, and symbolic). The results reveal that mathematics achievement (β = 0.49, p = .002) and algebraic-visual representation (β = 0.29, p = .03) were significant predictors of algebraic reasoning, whereas general ability (β = 0.11, p = .41), arithmetic-visual representation (β = 0.05, p = .65), and symbolic representation (β = 0.04, p = .71) were not significant predictors. The post hoc power analyses indicate our five-predictor regression model had power greater than 0.99 at an alpha level of 0.05.

Table 4

Summary of Regression Analyses

Predictors	*B*	β	*SE*	*t*	*p*
General ability	0.01	0.11	0.02	0.84	.41
Math Achievement	**0.15**	**0.49**	**0.05**	**3.29**	**.002**
Algebraic-visual	**0.30**	**0.29**	**0.13**	**2.22**	**.03**
Arithmetic-visual	0.05	0.05	0.12	0.46	.65
Symbolic	0.05	0.04	0.14	0.37	.71

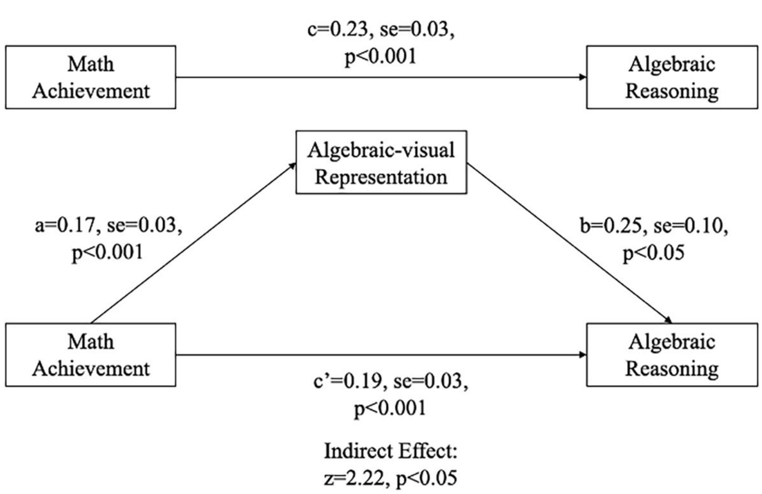
Note. Bold lines indicate statistically significant results.

Mediation Analysis

The results of the first mediation analysis, presented in Figure 2, indicated that there was a significant relationship between overall mathematics achievement and algebraic reasoning (path c). Further, generating algebraic-visual representations was found to be a partial mediator of the relationship between mathematics achievement and algebraic reasoning. Specifically, the indirect effect was significant, b = 0.04, 95% CI [0.01, 0.08], z = 2.22, p < .05.

Figure 2

The Model With Algebraic-Visual Representations as a Mediator

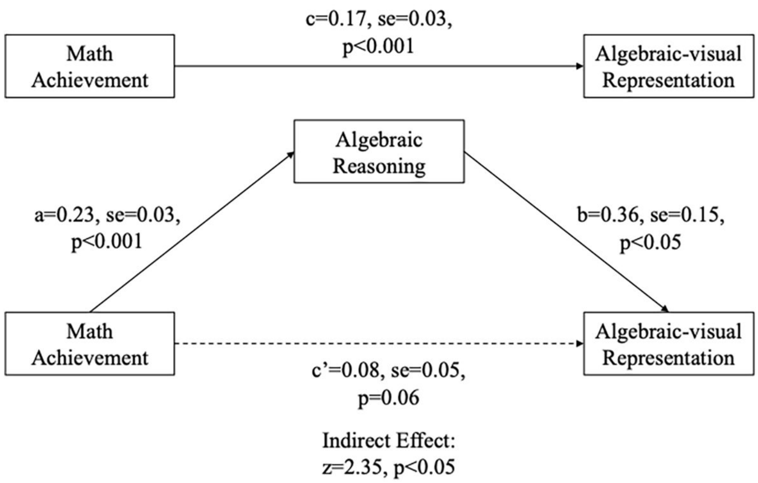


Note. Generating algebraic-visual representations was a partial mediator of the link between mathematics achievement and algebraic reasoning (p < .001).

In the second mediation analysis, there was a significant direct relationship between mathematics achievement and generating algebraic-visual representations, as shown in Figure 3. However, this relationship became non-significant when algebraic reasoning was included as a mediator, indicating full mediation. The indirect effect was significant, b = 0.08, 95% CI [0.01, 0.15], z = 2.35, p < .05.

Figure 3

The Model With Algebraic Reasoning as a Mediator



Note. Algebraic reasoning was a full mediator of the link between mathematics achievement and generating algebraic-visual representations (p = .06).

Discussion

In this study, we explored the link between algebraic reasoning and different self-generated representations (i.e., algebraic-visual, arithmetic-visual, and symbolic) of algebraic equations and inequalities, following the solution of these problems and after receiving a prompt to construct a visual representation. While we examined the relationships between all kinds of representations and algebraic reasoning, we specifically focused on the relationship between students' self-generated visual representations and their ability to reason algebraically. Our findings shed light on the visuospatial aspect of algebraic reasoning and help to improve our understanding of the well-established correlation between visuospatial abilities and mathematical competence (Atit et al., 2022; Casey & Ganley, 2021; Geary et al., 2021; Xie et al., 2020).

Higher algebraic reasoning scores were associated with a greater number of accurate self-generated algebraic-visual representations of equations and inequalities, above and beyond general mathematics achievement and general cognitive ability. Performance on the latter likely reflects, at least in part, individual differences in crystallized intelligence (general knowledge) that, in turn, is influenced by fluid intelligence and working memory (Cattell, 1987; Deary et al., 2007). In any case, the core finding is that domain-specific algebraic knowledge is related to using algebraic-visual representations of equations and inequalities above and beyond the contributions of domain-general abilities, as found with younger students (Geary et al., 2017; Lee & Bull, 2016). As noted, algebraic-visual representations of algebraic concepts might be an integral component of algebraic reasoning, suggesting not only that the latter influences the ability to generate the former but also that the link is mutual. The reciprocal relationship between the two can be illustrated through Bloom's Taxonomy.

Recall, Bloom's Taxonomy is a framework for categorizing learning outcomes into six stages: remembering, understanding, applying, analyzing, evaluating, and creating (Anderson et al., 2001). The construction of visual representations may fall into the application phase, where students integrate and transform fundamental algebraic concepts into a visual representation. The ability to generate accurate visuospatial representations might require solid algebraic reasoning. Once enhanced, self-generated visuals can facilitate the development of more robust algebraic reasoning, allowing the reorganization of problem information, abstraction of problem characteristics, and accurate representation of quantitative relationships (van Essen & Hamaker, 1990). That process could, at least, partially explain the well-documented relationship between spatial and mathematical abilities (Casey & Ganley, 2021; Geary et al., 2021; Halpern et al., 2007; Mix, 2019). However, the correlational nature of the data necessitates future longitudinal studies to confirm these hypotheses.

It is also worth mentioning that the algebraic-visual representations generated by the students were like the visualizations commonly used in algebra instruction, which suggests that students who received higher-quality or more advanced algebra instruction may have been exposed to these visualizations through instructional methods. However, this is not likely to be a systematic bias because the same curriculum and similar instructional materials were used in all schools attended by the students in this study. The key finding is that even with similar instruction and materials, students still differed in their ability to integrate conventional algebraic visual representations into their understanding of equations and inequalities.

The contents of self-generated visual representations also provided valuable insights into unique aspects of algebraic processes. The link between algebraic reasoning scores and self-generated algebraic-, rather than arithmetic-, visual representations of equations and inequalities aligns with differences in arithmetic and algebra processing, referred to as the cognitive gap between the two problem-solving approaches (Herscovics & Linchevski, 1994). This distinction might be analogous to the difference between schematic and pictorial visuals and their effects on solving mathematical word problems. Schematic visuals are more beneficial for word problem performance, while pictorial representations are not as effective (Edens & Potter, 2008; van Garderen & Montague, 2003). Similarly, algebraic-visual representations, as opposed to arithmetic visuals, may enhance problem-solving performance in algebra. Schematic visuals can represent relations among quantities embedded within a story context in word problems, and algebraic-visual representations can better represent the generalized patterns of relations in the problems to be solved and capture continuous relations presented in algebraic equations and inequalities in ways that arithmetic-visual representations cannot.

Arithmetic visuals cannot represent more abstract numbers, such as irrational numbers that do not have a physical analog (Patel & Varma, 2018). For example, there could not be a π amount of a ball. While some students were able to provide accurate arithmetic models that illustrated multiple scenarios using different values for the variable (for example, see Table 3), these models still had this deficiency. For example, even with expansion, these models were unable to represent more abstract numbers, such as irrational numbers, that do not have a precise physical analog. Two students who provided arithmetic visuals acknowledged that it was not possible to visually represent certain types of numbers. For example, one student stated that they could not visually show $\sqrt{3} + 1$ apples, indicating that they were unable to generate an algebraic-visual representation for this specific problem. Although one student was able to generate an algebraic-visual representation for the subsequent two problems, the other student was not able to produce an algebraic-visual representation for any of the three problems, suggesting that their ability to use these representations was still developing.

The algebraic-visual representations appear to become an integral part of students' algebraic reasoning once they have a strong understanding of the generalized features of algebra, including the concept of variables. However, researchers suggest that there is no direct transition from arithmetic to algebraic reasoning, and instead, students have access to two types of representations (Coles & Sinclair, 2019; Roth & Hwang, 2006). Students have firm arithmetic-visual strategies for arithmetic reasoning, and many eventually develop algebraic visualization for algebraic reasoning. According to Siegler's (1998) strategy choice model, both forms of representation are available for use and compete for expression based on the problem context and goals. Therefore, the goal of instruction is not to replace arithmetic-visual with algebraic-visual representations but to build students' ability to generate the latter and determine the appropriate contexts for their use.

Furthermore, accurate initial responses, indexed as solid algebraic reasoning skills, involved efficient symbolic usage. Therefore, the robust association between self-generated algebraic visuals and algebraic reasoning also suggests that students generating algebraic-visual representations are typically adept at using conventional symbolic algorithms flexibly. In line with that, a negative relationship was observed between algebraic-visual and symbolic representations. As students generated more algebraic-visual representations, they tended to prefer producing symbolic representations less. The pattern is consistent with prior studies of representational flexibility (Deliyianni et al., 2016) and with Siegler's (1996, 1998) overlapping waves and strategy choice model, which suggests that various problem-solving techniques are employed adaptively depending on the demands of the problem.

In summary, students with a solid understanding of algebra can flexibly produce symbolic representations to solve equations and inequalities but may inhibit their symbolic repertoire in favor of algebraic-visual representations, particularly when prompted to use visuals. However, some students who correctly use standard symbolic representations may not provide an algebraic-visual representation, even when prompted to do so. It is unclear why these students do not use algebraic-visual representations, but one possibility is that they do not have strong visuospatial abilities, which could make generating visual representations more challenging compared to using symbolic manipulations. This would be an interesting avenue for future research.

Educational Implications

The results of the study suggest several educational implications for teaching algebraic reasoning. The findings reveal that use of algebraic visuals is more advanced than algebraic reasoning, which seems to emerge after students have some foundational algebraic knowledge. When such foundational knowledge is established, visualization appears to enhance algebraic reasoning. Studies show that providing training to enhance students' visualization skills leads to an improvement in their mathematical performance (Hawes et al., 2022; Mix et al., 2021). Hence, fostering the visual component of algebra might be beneficial for students.

Students may not spontaneously generate algebraic visualizations but may need to be prompted to do so. It is important for teachers to recognize that individual differences in students' mental models of algebraic equations and inequalities may affect their ability to generate these visualizations. For students who can effectively use the provided algebraic visuals (in textbooks and during instruction), instruction and practice in generating their own visualizations may be beneficial. However, for those who struggle with using the provided visuals, instruction on interpreting them and generating their own may be beneficial. It might be the case, however, that some students have difficulties generating visual representations, even with instruction, and for them a focus on efficient use of standard symbolic problem-solving strategies might be preferrable, but this is unclear at this point.

Intervention studies have shown that using manipulatives and pictorial representations of quantities and variables, with a gradual transition to the conventional algebraic form, can be effective in teaching algebra (Long et al., 2021; Satsangi et al., 2016, 2018). An innovative approach to this method would be to prompt students to construct self-generated visuals after becoming familiar with provided manipulatives and pictorial representations. This technique has the potential to enhance students' algebraic reasoning in several ways. Constructing their own visual representations of a problem can reduce the demands on working memory, ultimately leading to greater problem-solving efficiency. This is supported by cognitive load theory, positing that decreased cognitive load is associated with increased performance (Chandler & Sweller, 1991).

Self-generated visuals might also provide insight if students fall back on using arithmetic visual representations. Prior research suggests arithmetic representation use can interfere with algebraic representation use; that is, competence with arithmetic-based representations will not automatically facilitate and may impede the learning and use of algebraic representations (Byrd et al., 2015). Thus, teachers need to carefully determine when and how to present arithmetic and algebraic visual models, as well as recognize the differences between the two and the deficiencies in using arithmetic visuals to represent algebraic problems. One approach to gaining insight into students' understanding of problems is to prompt them to represent problems visually, as this can provide helpful information that guides the next steps in instruction.

Limitations and Conclusion

The study had a few limitations, such as the fact that students could not attend regular in-person classes due to COVID-19, which may have affected their responses to the questions. Although we were unable to test if the differences in LGS scores between our sample and the entire population of test takers were statistically significant, this is likely not a significant issue given the strong correlations observed between their LGS mathematics scores and their performance on our tasks. Additionally, the data collection was online, which may produce different responses than an in-person interview. The study was also correlational, so future longitudinal and experimental research is necessary to understand the nature of the association between different visuospatial representations and algebraic reasoning and make causal inferences. Furthermore, the number of questions was limited, so the generalizability of the results to all areas of algebra needs to be further investigated. For instance, subtopics like quadric equations with a visuospatial component are worth examining.

Despite these limitations, this study extends our understanding of variation in how students think about and represent algebraic equations and inequalities as related to their ability to reason algebraically. The study also provides valuable insights into the link between visuospatial and mathematical abilities. Specifically, it suggests that competence with algebraic reasoning is dependent, in part, on the ease with which visuospatial systems can be used to represent algebraic relationships above and beyond the contributions of domain-general abilities and general mathematics achievement.

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Ethics Statement: This study had ethical approval from Boğazici University (IRB# E-84391427-050.01.04-18335/ 2021-31).

Data Availability: For this article, a data set is freely available (Ünal, 2023).

Supplementary Materials

The Supplementary Materials contain the data and R code for this study (for access see Index of Supplementary Materials below).

Index of Supplementary Materials

Ünal, Z. E. (2023). *Supplementary materials to "Visual and symbolic representations as components of algebraic reasoning"* [Research data and code]. OSF. https://osf.io/9f2de/

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Appendix

Table A1

Algebraic Reasoning Coding Rubric

	Scoring		
	2 points	**1 point**	**0 points**
Questions	**Providing an accurate answer through an accurate general statement.**	**Providing a partially accurate answer or an answer through substitution. The answer might have missing values, such as irrational numbers.**	**Completely wrong statements; no answer.**
What values of x make the following statement true: 5x is less than 30?	-… (the student's written work: "x<6") I wrote 5x is less than 30. So then, I divided both sides by 5. x was less than 6. It is valid for any value less than 6.	-… (the student's written work: "5,4,3,2,1,0,-1,-2,-3,-4,-5…") When we solved the equation, it asked xs less than 30. It asked about 5xs less than 30. It wanted the xs. It can be 1,2,3,4,5 for positives. It becomes equal; if it is 6, it becomes equal, it asks us for the smaller ones. In the continuation, it meets all the negative numbers, which can be small.	-… (the student's written work: "x = 5") Miss, it says here 5x is less than 30, it says for which value of x, it is correct. It says the value of x here, I don't quite understand it. Since 5x is a small number, of course it is less than 30
What happens to the value of y as x increases in value give the equation y = 2x + 3?	-… (the student's written work: "y-2x=3") When x increases, that is, when the subtracted number increases, this number also increases so that the result is the same, and the initial number also increases. Therefore, the value of y is also affected by increasing.	-… (the student's written work: "y=2x+3, x=1, y=5, x=2, y=7") If x increases, y also increases. I wrote an equation. So, I wrote the equation by giving a value. When I give 1 to x, 2x, 2 times 1 is 2 plus 3, y becomes 5. Then, I increased x. I gave 2 to x. 2 times 2 is 4, plus 3 is 7, which is y. So if x increases, y also increases.	-… (the student's written work: "y=2x+3, x=2, y=3") Here, miss, I didn't do any calculation, but I just wrote and thought. So, it says y equals 2x plus 3, miss, if x is 2, for example, y is 3, which one would increase uhm, if x increases continuously, I think y decreases, that is, I think it decreases.
Ben said, "w + 3" is less than "5 + w." Circle one: Always true Sometimes true Never true	-… (the student's written work: "2>0, w+2>w, w+5>w+3") First, it's always true because I know that 2 is greater than 0. If so, if I add w to each side of the equation, w plus 2 is greater than w, so w plus 0 is. If I add 3 to both sides of this inequality again, w plus 2 plus 3 uhm is greater than w plus 3. That is w plus 5 is greater than w plus 3.	-… (the student's written work: "-2+3=1, 5-2=3") I say uhm w, both of them, it gave as w, so it gave as the same letter, I naturally thought of it as the same number in this question. If I give 6 to both of them, I will get 9 from 6 plus 3, and 11 from 6 plus 5, that means 5 plus w is greater than w plus 3. So, the sentence he said is correct. Then, I tried 0. Because it can be thought of as different from other numbers sometimes, if I give 0, it comes as 3 from 0 plus 3 and 5 from 5 plus 0. Again, it meets the sentence. Then, I thought of negative numbers, so it could be a negative number. What number I gave, I gave -1. -2 plus 3 is 1 and on the other side, I think it means -2 plus 5 is 3, it meets the sentence. That's why I said always true.	-… Miss, I couldn't understand this question.

Note. Students' explanations were translated from Turkish to English.

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