Theoretical Contributions

A Tale of Two Researchers: Commonalities, Complementarities, and Contrasts in an Examination of Mental Computation and Relational Thinking

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Abstract

This paper describes a research collaboration between an educational psychologist and a mathematics education researcher, namely a didacticien des mathématiques. Our joint project aimed to explore the mental computation strategies of preservice teachers in an elementary mathematics methods course and to investigate the relationship between mental computation and relational thinking. The primary objective of the paper, however, is to go beyond the data and their interpretation. We describe the commonalities, complementarities, and points of contrast that emerged between us as researchers who hail from different disciplines, but who have the same overarching interests in mathematical thinking. In particular, we untangle issues we encountered during our collaboration related to our research questions, methodologies, and epistemological stances. We detail the ways in which we navigated these issues in the context of the research and describe what we learned about our own disciplinary perspectives and each other's. We conclude by discussing what our story offers as a means of reflecting on our individual fields and potential interactions between them.

Keywords: educational psychology, mathematics education, didactique des mathématiques, disciplinary perspectives, epistemology, mental computation, relational thinking

This paper describes a collaborative effort between two researchers from different fields with common interests in the nature of mathematical thinking. One of us, an educational psychologist, and the other, a mathematics educator (more precisely, from the field of *didactique des mathématiques*[i]), together explored the nature of mathematical thinking in preservice elementary teachers in a three-week mental computation unit. Our goal for this paper, however, is to go beyond a standard research report. We aim to describe our interactions during the various phases of the project, highlight contrasting and complementary issues that emerged, and illustrate how the collaboration shaped each other's approaches to the object under investigation.

We view this paper as a case study of what others have portrayed as differences between disciplines investigating the nature of mathematical thinking and development. In a recent issue in this journal, Alcock et

al. (2016) conducted a collaborative exercise with a number of researchers in the fields of mathematics education, psychology, and neuroscience to identify themes and key research directions that could serve to unify the three disciplines. They presented somewhat caricatural profiles by describing typical backgrounds and activities of the researchers from the three fields. According to the authors, mathematics educators come from teaching or mathematics backgrounds and conduct qualitative research on individuals' mathematical understandings from the early years to that of expert mathematicians. Psychologists who study mathematical cognition are often trained in experimental or cognitive psychology and conduct experimental work on basic processes underlying mathematical thinking.

Gauvrit's (2012) analysis supplements these depictions. He argued that a primary difference between the disciplines is that studies conducted by *didacticiens des mathématiques* are grounded in mathematics, which can be explained by their initial training with and in mathematics. Thus, *didacticiens des mathématiques* study mathematical thinking by taking the structure of mathematics into account and the mathematical objectives underlying educational actions. Historical or sociological layers are not necessarily discarded; when taken into account, these layers are also grounded in mathematical explanations and rationales.

Gauvrit argued that psychologists have a different focus; they often aim to uncover general processes in mathematical thinking, such as the role of working memory and arithmetic fluency, or to examine specific learning disabilities (e.g., dyscalculia). His summary yields an assessment each field has of the other: Psychologists underestimate the complexity of mathematics and *didacticiens des mathématiques* underestimate the complexity of the student. Gauvrit himself admitted the caricatural nature of his depiction, since nothing prevents the psychologist from taking mathematical elements of the teaching and learning situation into account, and the *didacticien* can concede to students' difficulties. The point, rather, is that the research endeavors of each field are grounded in disparate disciplinary rationales.

Gauvrit also underscored methodological differences. Whereas many psychologists conduct experimental studies to maximize internal validity, *didacticiens* have no training in experimental methods. Their approach to research is mainly theoretical, often grounded in *a priori* and *a posteriori* mathematical analyses of classroom situations (e.g., Artigue, 1988, 2015), and their data sources are based on observations of these situations. True experiments, in the psychological sense, rarely, if ever, happen in *didactique* studies. Under these conditions, then, it would be difficult to find two more incongruent approaches to conducting research on mathematical thinking.

As researchers, we could affiliate ourselves with these caricatures of psychology and mathematics education/ *didactique des mathématiques*. Our purpose in this paper is not, however, to argue that we "belong" in one category or the other or to paint revised prototypes of educational psychologists or *didacticiens des mathématiques*. Nor is it our aim to argue the relative merits of different research traditions and disciplines. Rather, our goal is to answer the call of this special issue by describing an authentic, if idiosyncratic, collaboration to reflect on possible points of complementarity and contrasts between our two fields. As such, our intention is to offer an orientation: something to work with and reflect on as a community of scholars. We aim to demonstrate how, despite clear contrasts between our interests and approaches, dialogue and collaboration is not only possible, but fruitful and enriching for an increased mutual understanding of, and sensitivity toward, each other's disciplines and enhancement of our own.

The Beginning of the Collaboration and Initial Emerging Issues

The research team consisted of the two authors of this paper and their two doctoral research assistants. In this section, the backgrounds and research foci of the two authors are described, which is then followed by a description of the genesis of the collaborative project.

The Research Team

The first author (henceforth referred to as HPO) is an educational psychologist whose research objectives are to understand the characteristics of instruction that promote (a) the appropriate use and interpretations of mathematical representations, and (b) the development of conceptual and procedural knowledge in instructional contexts. Her research is influenced by cognitive psychology, but her focus is on the application of cognitive theories to domain-specific accounts of how children learn mathematics in educational settings. Consistent with the historical traditions of educational psychology (Derry, 1992; Mayer, 1992, 1993), HPO most often collects experimental data using methodologies that employ random assignment to conditions, sample sizes that meet minimum statistical requirements, several outcomes measures, and quantitative analyses. This said, HPO's research is not always experimental. Depending on her purposes, she entertains questions that are best answered through descriptions about how students work through particular tasks or how they interpret specific mathematical representations. These descriptions serve to provide teachers with insight on children's thinking in particular contexts, assist to explain or illustrate quantitative results, or act as a springboard for future experimental research.

The second author (JP) works in a mathematics department and teaches future teachers of mathematics. As a *didacticien des mathématiques*, he focuses heavily on mathematics as a discipline and its specificity in the teaching and learning of mathematics. His research interests center on problem-solving processes and the mathematics that emerges in mental computation environments. His work addresses how mathematical and epistemological issues bear on mathematical cognition: Studies of cognition for JP are linked to epistemological questions about the nature of mathematics and its development (as much for mathematics as a discipline as for the person engaging in the mathematics; see Proulx, 2017a). Therefore, the data JP collects are often in the form of classroom observations of students' activity, because they offer unique opportunities – being in real-time and in action – to investigate and conceptualize the nature of problem solving.

The Beginning

The impetus for the collaboration came from one incident in a mathematics methods course HPO taught in an undergraduate elementary teacher training program in Canada[ii]. Her goal for the students was to expand their repertoire of mental computation strategies. In one class, HPO was explaining the compensation strategy by using 500 × 4 as an example. She demonstrated the strategy "one factor gives to the other" (Parker & Baldridge, 2004) and wrote 500 × 4 = 1000 × 2 on the board. At this point, a student raised her hand and said, "but I don't understand: 500 × 4 is not equal to 1000."

It became apparent to HPO that this student had difficulty grasping the meaning of the equal sign, at least in this specific context. HPO *hypothesized* that the student held what has been called an "operator" view of the symbol (Kieran, 1981; Seo & Ginsburg, 2003; Sherman & Bisanz, 2009) – a signal to compute rather than a representation of a relation between 500 × 4 and 1000 × 2. In this specific case, HPO entertained the possibility

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that this student's apparent misconception of the meaning of the equal sign impeded her ability to reason relationally about the equation. As a result of this classroom exchange, HPO wondered about the possible relationship – correlational or causal – between mental computation and relational thinking. She was particularly drawn to explaining such a relationship and whether it could be leveraged to support the development of relational thinking in learners of mathematics. With this in mind, HPO contacted JP to mount a collaborative research project around this problem.

Perspectives and Objectives

Through discussion, it soon became apparent that HPO and JP approached the problem in different ways. In this section, these perspectives are described, as are the objectives formulated by the authors as a result of their different approaches.

A View From Educational Psychology

The conceptualization of relational thinking offered by Carpenter, Levi, Franke, and Zeringue (2005) and Jacobs, Franke, Carpenter, Levi, and Battey (2007) served as an apt theoretical frame for HPO's hypothesis. Jacobs et al. defined relational thinking as a form of reasoning that entails looking at numbers and expressions holistically and noticing relations among them. Within this perspective, relational thinking is viewed as a form of algebraic reasoning that provides a conceptual foundation for the formal study of algebra (Carpenter et al., 2005). To illustrate, consider again the equation 500 × 4 = 1000 × 2. One way to determine the truth value of this equation is by computing the values on either side of the equal sign and then comparing the two amounts – e.g., 500 × 4 is 2000 and 1000 × 2 is 2000 – concluding that the equation is true. Such a strategy could be considered an example of "computational thinking" (Stephens, 2006) and can be completed strictly procedurally, with little understanding of the meaning of the quantities and operations involved. In contrast, thinking about the equation relationally entails reflecting on the numerical relationship between both sides of the equal sign. For example, seeing that 1000 is twice 500, it would need to be multiplied by a number that is half of 4 (on the left side) to make the equation true, invoking compensation as a way to make the amounts on each side of the equation the same.

The illustrations of computational and relational thinking in the above example are both dependent on a view of the equal sign as "the same as." According to Jacobs et al. (2007), there is more to relational thinking than a conceptual grasp of the symbol, however. While the equal sign features centrally in many descriptions of relational thinking (Freiman & Lee, 2004; Knuth, Stephens, McNeil, & Alibali, 2006; McNeil, 2014; Molina, Castro, & Mason, 2008), it is also contingent on a conceptual knowledge of numbers, operations, and number properties, all foundational to algebraic reasoning. In this example, the connection between relational thinking and algebraic reasoning is apparent because the same conceptual understandings undergird the procedure for solving for x in $500x = 1000 × 2$. An expanded conceptualization of relational thinking and the theoretical overlaps with mental computation is offered in the Appendix.

A View From Didactique des mathématiques

The generation of a hunch about the explanation of a psychological phenomenon – in this case, one that is closely tied to teaching practice – was HPO's entry into the research. JP had a different entry, however: As a *didacticien des mathématiques*, educational problems of this sort, or the search for causal links, are seldom addressed. When HPO introduced the research idea to JP, he first engaged in problematizing the links between

500 × 4 = 1000 × 2 and 500x = 1000 × 2. For him, more was to be said about the mathematics at stake. Although he agreed on the surface about the connection between the two equations, in his conceptualization, arithmetic and algebra do not share the same roots and the transition from one to the other is complex because *they engage in different mathematical processes* (Bednarz & Janvier, 1996). In short, an algebraic equation requires looking for all the values of x that would make the equation true, and the solver has to accept – that is, operate with – unknown values in similar ways as with known values (i.e., givens). This results, for example, in *isolating* the x on the left side of the equation by dividing both sides by 500 (Filloy & Rojano, 1989), or considering *equivalent equations* (see Arcavi, 1994), where one operates on the complete equation to find ones that are easier to solve (500x = 1000 × 2 becomes 1000x = 1000 × 2 × 2 = 4000 leading to x = 4). If, on the other hand, one wishes to make the connection between 500 × 4 = 1000 × 2 and 500x = 1000 × 2, this leads to viewing the x in 500x = 1000 × 2 not as algebraic, but rather as a place-holder, as commonly seen in the elementary mathematics curriculum (e.g., □ + 6 = 9, what is □?). The place-holder conception requires an entry where operations are arithmetically "undone," in terms of *unwinding* (Hewitt, 1996; Nathan & Koedinger, 2000) to arrive at a value for x: "I undo the operations made on x: x was multiplied by 500, so 1000 × 2 is to be divided by 500 too." As Filloy and Rojano (1989) explained, when using this entry, "it is not necessary to operate on or with the unknown" (p. 20), as it becomes a series of arithmetical operations performed on numbers, avoiding the usage of algebraic reasoning altogether.

Such distinctions in the approaches to "the problem" highlighted other differences about the contexts in which each researcher's objectives were grounded[iii]. For HPO, the means with which to explore whether the preservice teachers indeed *learned* – i.e., improved in their relational thinking – was to examine the cognitive processes that characterized Carpenter et al.'s (2005) theoretical framework. Central questions for HPO were: Did the preservice teachers engage in computational strategies to determine the truth value of mathematical equations or were they able to think about the relationships among the quantities in the equations? Can we find evidence to support the claim that the mental computation intervention facilitated their relational thinking? If the research generates such evidence, what can be said of the theoretical overlap between mental computation and relational thinking? Such questions make evident that HPO was driven by the search for evidence that would make recommendations for teaching practice credible.

In contrast, HPO's experience with the preservice teacher would not have led JP to question the links between relational thinking and mental computation. Rather, some of the questions he might have generated are: "What sorts of 'equal signing' is happening in a mental computation session?"; "What mathematical meanings are invoked during the mental computation sessions?"; "What sorts of mathematical understandings of the equal sign are the students developing and interacting with?"; and "What sorts of obstacles (epistemological, didactical, cultural, historical; Brousseau, 1983, 1989) are the students facing during the mental computation sessions?". These questions focus directly on, and emerge from, the nature of the mathematics itself and of students' mathematical understanding.

A Problem-Solving Perspective

JP was intrigued by HPO's proposal, but his interests in it were motivated by a continued need for investigating and conceptualizing problem-solving processes in mental computation environments. Most studies concerned with analyzing students' mental computation strategies are (implicitly) based on the assumption that has often been termed the "toolbox metaphor" (or the "selection-then-execution" hypothesis). The toolbox metaphor implies that when solving a problem, students choose a strategy (i.e., a tool) from a group of predetermined

strategies they have acquired or created in the past (i.e., the toolbox). The subsequent data analysis often becomes centered on outlining and cataloging the various strategies used, and, in turn, instruction becomes a matter of explicitly teaching these strategies so that students use them in appropriate ways.

Recently, however, the toolbox metaphor has been questioned by a number of researchers, who illustrate that there is more to solving mental computation tasks than the "simple" re-using of already predetermined strategies. Threlfall (2002, 2009) insists rather on the organic emergence and contingency of strategies in function of the tasks and the solver himself (e.g., what the solver understands, prefers, knows, has previously experienced, has confidence in; see Butlen & Peizard, 1990). He explains that strategies vary as much as problems do, so decisions about the "right choice" or the "good method" to choose "cannot be arrived at in advance of actually moving forward into calculating the problem" (Threlfall, 2002, p. 38). Lave's (1988) situated cognition perspective is also aligned with these views; she conceives of mental strategies as flexible emergent responses, adapted and linked to a specific situation.

JP's work is aligned with this perspective and draws on the enactivist theory of cognition to study mental computation solving processes (see, e.g., Proulx, 2013b). *Enactivism* refers to a theory of cognition that views human knowledge and meaning-making as processes understood and theorized from a biological standpoint (e.g., Maturana & Varela, 1992; Varela, Thompson, & Rosch, 1991). By adopting a biological point of view on knowing, enactivism considers the organism as interacting with and in an environment: A student evolves with and in her environment in a connected and adapted fashion, where her strategies or mathematical solutions are not necessarily optimal, but are functional (i.e., "good-enough," Zack & Reid, 2003, 2004), with and in a context that is itself evolving under the influence of the solver. Mathematical strategies are not considered as existing *a priori* (or *a posteriori*) of the moment they appear. They are the real-time product of interaction between the solver and her environment, directly and continually emerging from and influenced by both, generated on the spot by the solver for the task at hand. Thus, HPO's proposal was seized by JP as an occasion to delve more deeply into these processes.

Defining the Research Objectives

These initial discussions led to two specific local objectives for the research. HPO's intention was to collect evidence that might demonstrate a relationship between mental computation and relational thinking, which could provide evidence of learning. JP's objective was to capture in action and investigate solving processes generated in the context of mental computation tasks so that their emergent mathematical nature could be further investigated. Through this collaboration, both HPO and JP followed and addressed each other's research approaches, techniques, and objectives, thereby generating an understanding, or curiosity, about how the differences in their disciplinary perspectives could enhance cross-fertilization of ideas and deepen their research.

Method

Gauvrit (2012) proposed that the main impediment to collaboration between psychologists and *didacticiens des mathématiques* is methodology. Indeed, in our case, prominent points of contention most often revolved around method, where conversations permitted *agreement* on certain methodological options, and *acceptance* of

others. This section focuses on the reasons for each decision related to data and design, and highlights issues marking significant disciplinary differences.

Design

HPO and JP needed to make decisions about how the intervention would be delivered and the tasks that would be used in the unit. In this section, the mental mathematics unit is described, followed by a discussion of key decision points related to tasks and instructional procedures.

Course Context and the Mental Mathematics Unit

The participants were 33 preservice teachers in the first of three required mathematics methods courses in an elementary teacher training program at a large, urban university in Canada. Six 45-minute class sessions were devoted to mental computation activities. In three of the sessions, students were asked to solve a number of arithmetic tasks mentally (e.g., 741 - 75; 184 ÷ 8). HPO read each task out loud, and students were given 20 seconds to arrive at an answer. They were not permitted to use any tools (e.g., paper and pencil, fingers) during the solving activity. When time was called, the instructor called on the students to describe their strategies, and at times invited them to the board.

During whole-class conversations about strategies, HPO's focus was to help students understand the underlying principles that justified each strategy (e.g., the meaning of multiplication, commutativity, distributive property) and to see the connections between different strategies for the same task. While HPO focused on such principles for instructional purposes (i.e., this was a portion of the course's curriculum and reflected in the assigned readings), she was also conscious of ensuring that the unit indeed fostered appropriate mental computation strategies for research purposes. For JP, this aspect of the unit was significant because the explanations and interactions that occurred in the session were, for him, the object of inquiry.

In the other three sessions in the unit, students worked individually on practice problems (e.g., exercises) delivered on worksheets. In these practice sessions, the students were required to solve tasks mentally and describe in writing the strategies they used. The worksheets were graded and returned to students with feedback.

Choice of Tasks

The tasks used in the unit were not chosen randomly. As instructor of the course, HPO aimed for tasks that were likely to generate specific strategies from the course readings (Parker & Baldridge, 2004). For her, this was required to investigate the theoretically hypothesized transfer to relational thinking. For JP, the chosen tasks needed to be grounded in mathematics, developed through and oriented by a *conceptual analysis* (a particular research tool from the French *didactique* scholars, e.g., Brousseau, 1998). In short, a conceptual analysis aims to draw out, among other things, the key understandings and main obstacles, difficulties, and common errors that could theoretically occur in relation to a specific mathematical idea or group of ideas, in this case numbers and the four arithmetical operations (see Proulx, 2017b, for an example of a conceptual analysis in the domain of statistical average).

As a way of illustrating the use of the conceptual analysis in the present context, consider the task 184 ÷ 8. This task was chosen by JP in particular because a simple decomposition of 184 into 180 and 4 would not permit a straightforward mental computation strategy (i.e., 180 ÷ 8 and 4 ÷ 8 do not yield whole numbers) and at the

same time permits a number of different entry points for mental computation. For example, the versatility of 184 ÷ 8 is related to the iterative nature of 8 as a divisor – that is, 184 can be divided repeatedly by all the factors of 8 (i.e., 8 can be thought of as dividing by 2 three times), yielding 184 ÷ 2 = 92, 92 ÷ 2 = 46, and finally 46 ÷ 2 = 23. In addition, 184 ÷ 8 is amenable to a number of different applications of the distributive property, namely that "If number *a* can be decomposed into parts and if each part is divisible by the number *b*, then the entire number *a* is divisible by the number *b*." For example, 184 can be decomposed in a variety of ways such that each addend is divisible by 8 (e.g., 184 can be decomposed into 160 + 24 or 80 + 80 + 24 or 80 + 104, with each part divisible by 8).

Instructional Procedures

There were a number of methodological decisions that needed to be made regarding the delivery of the mental computation unit. First, it was important to HPO to pinpoint the elements of the instruction that might be responsible for the differences in relational thinking after the unit was completed. To her, this was particularly critical because there was no treatment integrity built into the study design. As such, HPO had initially intended to "directly teach" specific strategies to her students by labeling them and explicitly describing how and when to apply them. For example, for a computation involving ÷ 5, a useful strategy would be divide by 10 and double. By directly teaching the strategies, HPO felt she could ensure that the instruction targeted mental computation – and as much as possible *nothing else* – so that the data would bear on its relationship to relational thinking.

In contrast, explicit instruction (or "telling") of specific strategies ran against the research goals of JP. To meet JP's objectives, the students needed to engage in whatever methods were meaningful to them (arising *in situ*; Lave, 1988; Varela et al., 1991; Varela, 1996), and to avoid having students merely pleasing the instructor by using what was shown beforehand (something akin to what *didacticiens des mathématiques* call the didactic contract; Brousseau, 1998; Sarrazy, 1995). The tension between control (from a design perspective) on one hand, and the investigation of authentic, emergent mathematical practice on the other, was a significant issue that needed to be resolved. HPO accepted to allow students to use strategies that were meaningful rather than reproduce "pre-fabricated" ones. She conceded that such reproductions might actually decrease the likelihood that mental computation actually took place, which would ultimately compromise her ability to draw conclusions about its connection to relational thinking.

Furthermore, when asked to solve mental computation tasks, children have been known to perform standard algorithms mechanically (Heirdsfield & Cooper, 2004; Sowder, 1992; Varol & Farran, 2007). Knowing this, HPO wanted to prevent the preservice teachers from doing the same and ensure that they engaged in mental computation that was driven by flexible mental transformations of quantities and mathematical expressions. As such, she specifically instructed the students not to use standard algorithms. JP accepted to limit the students' use of the standard algorithm, even if it could still be considered a mental strategy. He thus proposed a 20-second time limit for each mental computation task; his suggestion was informed by previous work, showing that reasonable time limits tend to encourage strategies that are idiosyncratic and personally meaningful (Butlen & Peizard, 1990).

Finally, HPO based the instruction on existing research in educational psychology revealing that the types of equations to which children are exposed can impact their conceptions of the equal sign, even with minimal teacher involvement (McNeil, 2008; Osana & Sokol, 2014). During the mental computation unit, she made certain that the equations written on the board were always in "canonical," or standard, form – that is, those that

have one number to the right of the equal sign and the operations only to the left of it (e.g., *a + b = c*; *a × b = c*). This was to eliminate exposure to non-canonical equations as a way to account for any potential differences in relational thinking after the unit; JP found this acceptable because it did not compromise the nature of his observational data.

Data Sources

The data came from two sources, chosen to address each researcher's specific objectives. A paper-and-pencil measure of relational thinking was administered before and after the unit, and observational field notes were collected during the unit.

Measuring Relational Thinking

Establishing a causal link between mental computation and relational thinking was the ideal objective for HPO. To establish cause, however, a researcher would need to collect evidence under specific design conditions. HPO's ultimate objective was to design a study that would eliminate as many alternate explanations for the results as possible[iv]. This was important for HPO so that she could claim, and hence *explain*[v], that it was in fact the focus on mental computation, and not something pre-existing or co-occurring, that was responsible for any improvement in the students' relational thinking. A "controlled randomized trial" would have served this purpose, but was not possible given the constraints present at the time of the research: There was only one section of the methods course available and not enough time could be devoted to the delivery of a control intervention to half the students. Thus, given such constraints, HPO strived to reduce the number of alternative explanations for the results rather than eliminate them entirely.

To gather data that could coordinate with the corresponding theory (Kuhn, 2010; Shavelson & Towne, 2002) about the theoretical relationship between mental computation and relational thinking, HPO and JP created the Relational Thinking Test (RTT), a paper-and-pencil instrument to assess the preservice teachers' relational thinking. JP did not require a measure of relational thinking to meet his research objectives, but he understood that this measure was required for HPO's part of the research and thus accepted to co-construct the measure. The RTT, shown in Table 1, consisted of six arithmetical expressions involving whole numbers. Each item required the student to "indicate whether the following number sentence is true or false" and to justify his or her response (see Carpenter et al., 2005).

Table 1

Items on the Relational Thinking Measure at Pretest and Posttest

Item	Pretest	Posttest
1	347 + 108 = 108 + 347	228 + 319 = 319 + 228
2	82 × 3 = 82 × 2 + 83	156 × 4 = 156 × 3 + 157
3	10000 ÷ 4 = 5000 ÷ 2	5000 ÷ 4 = 2500 ÷ 2
4	117 + 56 − 117 = 58	4508 + 124 − 4508 = 124
5	25000 − 7899 + 1 + 7899 = 25001	24780 − 893 + 1 + 893 = 24781
6	8607 × 56 = (60 - 4) × 8607	1350 × 48 = (50 - 2) × 1350

The items on the RTT were constructed to measure the students' ability to transfer what they learned from the mental computation unit to contexts requiring relational thinking. For example, consider item 2 on the posttest.

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To think about this item relationally, one may invoke (implicitly or explicitly) the distributive property: Four groups of 156 would be the same as three groups of 156 and an additional group of 156. Recognizing the resulting inequality by adding a group of 157 to the three groups of 156 would obviate the need to compute the amounts on both sides of the equal sign – the hallmark of relational thinking according to Carpenter et al. (2005). The psychological notion of transfer (e.g., Barnett & Ceci, 2002; Mestre, 2006) is important here. In HPO's sessions during the instructional unit, her goal was on helping students see how principles of arithmetic justified their mental computation strategies, but outside the context of judging the truth value of mathematical equations. Switching the context to one that would require relational thinking (at least as explicated by Carpenter et al., 2005) *by transferring the concepts learned during instruction* is a common approach used by educational psychologists to measure understanding.

Observations of Mental Computation Strategies

To investigate mathematics problem-solving processes in action and the mathematics students enact in their strategies, JP required data that captured how students solve mental computation tasks. Thus, observational data of students in the throes of mathematical activity were gathered. JP and the two research assistants attended the class sessions during which students solved and then explained their strategies to their peers. Data were taken in note form, where all three observers took detailed field notes of the strategies described orally by the students and the explanations that ensued. One of the assistants also took pictures of the students' work that remained on the blackboard after they had explained their strategies. The combination of the three sets of field notes, the photographs, and the on-the-spot short discussions among the team members after the sessions made a rich corpus of data[vi].

Data Analysis

This section offers descriptions of how the data were treated, coded, and analyzed. These descriptions serve to explain and justify how conclusions were drawn from the data collected on the project.

Relational Thinking Test (RTT)

HPO developed a rubric based on the frameworks of relational thinking outlined by Empson, Levi, and Carpenter (2011) and Carpenter et al. (2005) to code the students' justifications on the pre- and posttest items. The rubric consisted of three major categories: Relational thinking (RT); Computational thinking (C); and Other (OT). Relational thinking was operationalized as reasoning based only on number properties with no accompanying computation. Computational thinking was characterized by computing the amount on each side of the equal sign and comparing to see if the amounts were the same. Justifications in the Other category were those that were either irrelevant, incomplete, or that demonstrated incorrect use of mathematical properties or concepts.

Discussing the rubric with JP, however, highlighted yet another disciplinary difference, this time with respect to analyzing data on mathematical thinking. Whereas HPO was (initially) bound by Carpenter et al.'s (2005) framework, JP relied on epistemological analyses of the students' written justifications. More specifically, JP argued that in mathematics, understanding is *formalized* through representations, including numerals and other symbols. Furthermore, such representations are in fact explicit demonstrations or justifications *of* mathematical thinking (Hanna, 1989; Hersh, 1993). In this context, then, a written computation represented formally with symbols (e.g., an algorithm, a simple computation) may on the surface appear as if no relational thinking took

place, but may in fact serve as evidence of an explanation or justification of some underlying mathematical reasoning (relational or not). The rubric was thus expanded to include distinct sub codes for two instantiations of relational thinking – namely, a sub code [RT – No Computation] for instances of the "no computation" benchmark proposed by Carpenter et al., and a sub code [RT – Justification] to recognize cases where students used computation to support or explain a form of relational thinking.

Mental Computation Strategies

The contextual background against which the data were considered, grounded in issues of emergence and situativity, welcomed any idiosyncratic or even locally-functioning strategy offered by the preservice teachers during the unit. Embracing Plunkett's (1979) notion that mental strategies are fleeting, variable, flexible, active, and also local, tailored, and connected to the task at hand, the data analysis aimed to characterize the mathematical activity generated in this mental computation context, untangling the steps taken as students solved tasks and scrutinizing the nature of their problem-solving processes. Following Douady (1994), the intention was not to report on the learning that took place, nor to discuss the long-term effects of such problem-solving processes in other mathematical situations. Rather, JP aimed to gain finer understanding and conceptualization of these processes, both in terms of their functionality for the tasks at hand and the meanings created by the students.

The observational data were analyzed in a two-phase process. The first phase consisted of brief team meetings (JP, HPO, and the research assistants) after each instructional session to review and summarize the events that occurred during the session, focusing on students' strategies. The primary purpose of these meetings was to supplement the field notes with additional observations and insights. The team meetings also offered a first level of analysis that included the teacher educator's voice (HPO), which afforded interpretations of the events from a practitioner's perspective, and observations from other members of the research team on specific aspects of the session that merited attention. These discussions sharpened the team's understanding of solving processes, which were in turn used to interpret the observations in subsequent sessions. The second phase of the analysis consisted of repeated interpretive readings by JP of all field notes, after which the analyses were presented to HPO and the assistants. The resulting discussions centered on the relevance and nature of the analysis, which were strengthened through joint conceptualizations and interpretations.

Results and Findings

The following reports on both the results and the findings[vii] of the research. Each data set is presented separately (i.e., data on relational thinking and observational data on mental computation strategies) as well as the interpretations of each researcher's analyses.

Results: Assessing Improvement in Relational Thinking

Given that HPO was primarily interested in determining the increase in relational thinking from pretest to posttest, all justifications on the RTT that were coded C (i.e., computational thinking) and OT (i.e., other) were collapsed into one non-relational category (NR). This resulted in two possible codes for each item on the RTT: (a) RT (relational thinking), which included both RT sub codes, and (b) NR (non-relational thinking). Each participant was then assigned either an overall RT or NR code at pretest and posttest depending on the

proportion of RT codes assigned in total on the RTT at both time points. A conservative threshold of at least 5 out of 6 RT codes was used – that is, students who were assigned an RT code on at least 5 of 6 RTT items on the pre- and posttest were considered relational thinkers. The others were considered non-relational thinkers.

The proportion of relational thinkers at pretest was 48.5% (16 of 33 participants), whereas the proportion of relational thinkers at posttest was 78.8% (26 of 33 participants). A McNemar test found that the change in proportions of students becoming relational thinkers was significant, $\chi^2(1, N = 33) = 6.25$, $p = .012$.

Table 2 illustrates the nature of the observed change for two students, Brittany and Claude (pseudonyms), from before to after the mental computation unit. Their responses serve as exemplars of the types of computational and relational thinking described in the literature (Carpenter et al., 2005; Empson et al., 2011; Stephens, 2006). Before the unit, both Brittany and Claude were engaged in computational thinking by calculating the amounts on each side of the equal sign and then comparing them. After the unit, both students examined the expressions in their totalities, taking into account the quantities on both sides of the equal sign in relation to the operations involved. Brittany's work illustrates an application of the distributive property to reason about the quantities involved. Seeing that 156 × 4 would be the same as (156 × 3) + 156 allowed her to conclude that it could not be equivalent to (156 × 3) + 156 + 1. Claude's work illustrates intuitive knowledge of the inverse element for addition that for every real number a, $a + (-a) = 0$.

Table 2

Illustrative Examples of Responses on Selected Items From the RTT Before and After the Mental Computation Unit

| Student | Before Unit | | After Unit | |
	Item	Response	Item	Response
Brittany	True or False: 82 × 3 = 82 × 2 + 83	Student computed amounts on each side of the equal sign and compared numbers (246 ≠ 247).	True or False: 156 × 4 = 156 × 3 + 157	With no visible computations, the student wrote: "*This is false because on one side you are adding 4 sets of 156, on the other you are really adding 4 sets plus an extra 1. 156 × 4 + 1.*"
Claude	True or False: 117 + 56 – 117 = 58	Student computed amounts on each side of the equal sign and then wrote: "*False because when we do the addition 117 + 56, the result is 173. When we subtract 117 from 173, the remaining is 56 and not 58.*"	True or False: 4508 + 124 – 4508 = 124	With no visible computations, the student wrote: "*This is true because you are subtracting the same number that you added in the first place.*"

The implications of these results for practice are important, in HPO's view. As Carpenter et al. (2005) pointed out, the fundamental properties of arithmetic, which explain almost all procedural transformations in the school curriculum, are hidden behind the "notational elegance" of the standard algorithm. Too many students enter high school with procedural knowledge (i.e., knowing how to perform symbolic transformations) that is divorced from the conceptual rationales that give meaning to mathematical activity. Should mental computation be related to relational thinking, and especially if a causal relation is found in future research, one may conclude that engaging students in activities that increase their strategy flexibility, such as mental computation, lays the groundwork for genuine understanding in mathematics.

Findings: Observations and Conceptualizations of Mental Strategies

JP's intention is seldom to report on all the strategies *themselves*. Rather, he uses strategies generated by individuals as vehicles for extending and contributing to existing conceptualizations on strategy processes in mental computation. In similar ways to psychology, the intention of many *didacticiens des mathématiques* is theory building, but it is first and foremost to offer conceptualizations of, and fruitful distinctions about, mathematical thinking processes (Proulx, 2015). Therefore, the findings that follow are not results *per se*, but are orientations, ways of drawing out distinctions about problem solving in mental computation. The discussion below addresses such distinctions in the context of one specific strategy described orally by a preservice teacher, Amy (pseudonym). Amy's case is also well suited for illustrating issues about strategy generation and their emergence in relation to the task at hand.

For the task 741–75, Amy's work is shown in Figure 1. She explained her mental computation strategy orally and at the board in the following way:

a. 741 – 75 is like 700 – 75 + 41.

b. 700 – 75 is like having 7 dollars and subtracting 3 quarters. I am left with $6.25. And, 6.25 is six-twenty-five, so I add 41 to 625.

c. To do so, I do 5+1 is 6, 2+4 is 6, and I have 600, so 666.
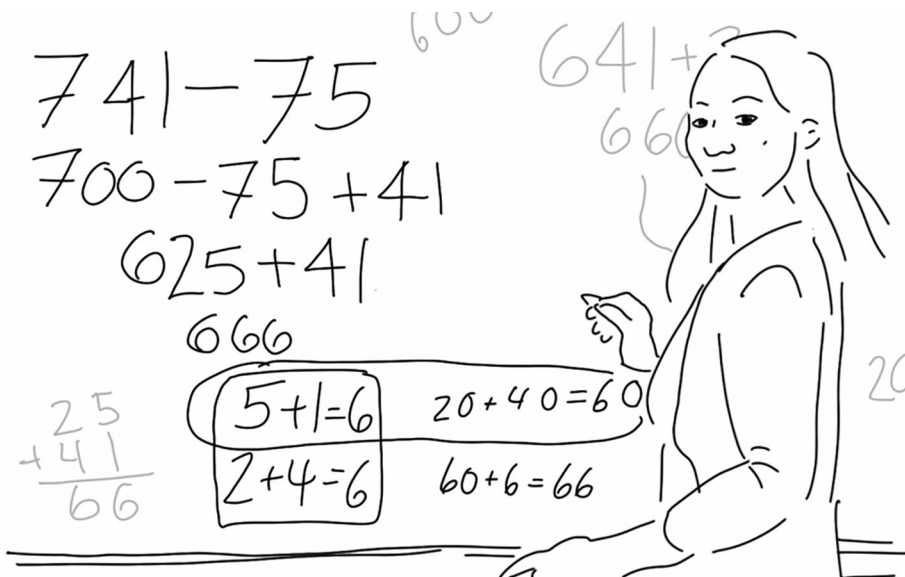

Figure 1. Amy's presentation of her strategy on the board.

Focusing on the nature of the strategy as emergent, but contingent on the task itself, one can argue that going for 700 is not a step that came out of the blue, as it is related to the fact that 75 can be easily subtracted from 700 compared to, for example, 67, which would be arguably more difficult. Attempting to take 67 from 700 would have likely produced a different strategy altogether. Again, the remaining steps taken by Amy could be seen as directly related to the task, or more precisely said, as a function of the numbers (and in relation to what the student can see as possibilities with those numbers). Furthermore, each step of Amy's strategy can be seen as a "new" task to solve, thus necessitating additional "ways in" that allow her to continue reasoning

through it. Specifically, step (a) in Amy's strategy is a way to "enter" the task, the problem, and the outcome of this step (having to compute 700 – 75) places her in front of another "problem to solve." Again, she must find a way to continue and thus opts for finding a money context ($7 minus 3 quarters) to carry out the computation. Again, the outcome leads Amy to another sort of "obstacle" to overcome, which is to find a way to compute 625 + 41, leading to a strategy of splitting ones, tens, and hundreds to add like units. Amy's strategy shows that solving 625 + 41 mentally is not "obvious" to everyone, but is instead dependent on the unique characteristics of the solver, including her understandings and ways of doing mathematics.

Of interest is the fact that this specific strategy was generated for *this* task. Amy opted for splitting 741 into 700 and 41, and then subtracted 75 from the 700. Although fitting, this is likely not a strategy Amy would have used if the work had been carried out with pencil and paper. In pencil-and-paper contexts, students often resort to the standard algorithm, working their way through the task in the way they are accustomed, mostly independent of the task given (e.g., its numbers). Indeed, algorithms are intentionally created and used to solve an entire class of problems regardless of the specific quantities involved, and their power is that they "work" every time (Bass, 2003).

In contrast, students' *first* intention in a mental computation context is to find a way to "enter" the problem. That is, when the task 741–75 was given, Amy's first action (or first "reaction with knowledge," to use Threlfall's, 2002, expression) was to find a way to solve it, to "find a way in." It is in this sense that the mental computation strategies generated by the solver are contingent on the task given, tailored to it, generated through it, and thus also emergent, developed on the spot when the task is received. As Threlfall (2002) explained, "each solution 'method' is in a sense unique to that case, and is invented in the context of the particular calculation – although clearly influenced by experience. […The strategy…] is not decided, it emerges" (p. 42).

Mental strategies are then conceived as unique, new, generated in the act of solving, and creatively produced in a problem-solving context[viii], a conceptualization epistemologically aligned with the nature of mathematicians' practice, defined as a creative and productive activity (e.g., Burton, 2004; Livingston, 2015; Lockhart, 2009; Papert, 1972). This alignment represents an illustration of how, for *didacticiens des mathématiques*, findings become epistemologically laden (Proulx, 2017a).

Final Remarks

Have you heard the one about the educational psychologist and the *didacticien des mathématiques*? An educational psychologist meets a *didacticien des mathématiques* and says, "we need to control the variables." The *didacticien* replies "hmm, but that's not easy since algebraic variables imply co-variation in a functional context"!

In this paper, we described a collaboration between an educational psychologist and a *didacticien des mathématiques* who investigated the mathematical activity of elementary preservice teachers. In our description, we deviated from the format traditionally used to report empirical results to instead highlight disciplinary commonalities as well as contrasts in our collaboration. Table 3 highlights a number of themes that captured these contrasts.

Table 3

Points of Distinction That Emerged Through the Research Collaboration

Themes	HPO's perspective (Educational psychologist)	JP's perspective (*Didacticien des mathématiques*)
Research Questions		
General research interests and objectives	Develop evidence to make credible recommendations for teaching practice	Make sense of the nature of the mathematics generated by students, solvers, and so on, through their activity
Local research interests and objectives	Explore the link between mental computation and relational thinking; Obtain pre- and post- data to test the effects of an intervention that is based on theoretical commonalities between the two constructs	Investigate problem-solving processes in mental computation environments; Develop more refined understandings of mathematical solving processes; Capture and analyze the mathematical strategies in action when doing mental computation
Methods		
Design of intervention tasks	Aligned with specific strategies hypothesized to foster relational thinking	Grounded in mathematics, designed through a conceptual analysis about numbers and operations
Mathematical content – equal sign	For algebraic preparation	No direct link between arithmetic and algebra; Engages in different mathematical domains (place holder vs. unknown/variable)
Role of mental computation unit	Intervention	Observational data
Instructional constraints	No standard algorithm and no presentation of the equal sign; Intervention targets mental computation and nothing else	To unleash solvers' natural strategies; No pre-guided strategies are given to ensure solvers' authentic engagement process
Data	Measure: written test administered before and after the intervention	Observational field notes of solving actions; Pictures of board as traces
Nature of the evidence	Aggregated data	Idiosyncratic strategies, from one person locally solving
Rubric development	Aligned with theoretical frame of relational thinking	Aligned with the mathematical historical development in the discipline in relation to sense making and symbolizing
Reporting	Results; Statistical analyses	Findings; Long descriptions, interwoven with theoretical assertions
Epistemology		
Views on knowledge	Positivist/empiricist	Postmodern
Epistemological goals	Generalize results on links between mental computation and relational thinking	Generate ideas about mathematical solving processes

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On Disciplinary Differences

Disciplinary differences that surfaced within our collaboration included research questions, methodological choices, and epistemology. First, our research questions underlined significant discrepancies in the way we each entered the research project, reminiscent of Toulmin's (1977) depiction:

> If we mark sciences off from one another […] by their respective "domains," even these domains have to be identified not by the types of objects with which they deal, but rather *by the questions which arise about them* […] Any particular type of object will fall in the domain of (say) "biochemistry," only in so far as it is a topic for correspondingly "biochemical" questions. (p. 149, our emphasis)

Second, as Gauvrit (2012) would have predicted, our approaches to data collection and the ways in which the data were analyzed and interpreted were decisively different, even divergent, from one another. Third, an arguably insurmountable difference was related to our underlying epistemologies – that is, the nature of our beliefs about what it means to "know" something as researchers. In a caricatural way, although possibly accurate in some cases, Berch (2016) referred to epistemological differences between cognitive psychologists and mathematics educators: "The objectivist/mechanistic/positivist epistemology that purportedly undergirds experimental cognitive psychology is viewed as inconsistent with if not antithetical to the constructivist epistemology that Thompson asserts is 'taken for granted' by contemporary mathematics education researchers" (p. 45).

In somewhat analogous, but perhaps more nuanced ways, Berch's epistemological dichotomy featured in our collaboration as well. More precisely, HPO assumed a positivist/empiricist orientation to the research through the coordination of theory and data, and JP a postmodern one, focused on emergence, connectedness, and situativity of events. The collaboration provoked confrontation in our disparate epistemological perspectives. Such tensions were salient as we grappled with issues on how the data would grant us understandings about mathematics, mental computation, and relational thinking, and the ability to explain them — that is, to generate scientific knowledge for our fields of study. Ultimately, for HPO, her objectives required data that would demonstrate at least a correlation between two constructs that were empirically observed: a necessary, but not sufficient, condition for establishing a causal connection. For JP, on the other hand, the intention was mainly to generate new ideas and new distinctions that would push forward or trigger the field's knowledge about the themes under study.

On the Nature of Collaborating

Regardless of our seemingly discordant research questions, methodologies, and epistemological paradigms, our interactions permitted *agreement* on a variety of issues and *acceptance* of others. The collaboration and the opportunity we took to write about it have been important learning events for both of us as researchers. We broadened our understandings about our respective fields in ways that would likely not have occurred had we remained in our individual silos. Indeed, being confronted with the other's objectives, methods, and "voices" prompted us to refine our own.

With this in mind, then, we do not concur with Berch's rather gloomy outlook on the apparent disciplinary rift between mathematics education and cognitive psychology – or educational psychology in our case. More concretely, we argue that there are more productive responses to what he calls a "developmental disconnection syndrome" between the two fields than a facile application of psychology to practices in mathematics education

research. Some 45 years ago, Piaget maintained that psychological methods, at least those that existed at the time, are not by themselves useful when they are "parachuted" (our term) into a complex setting, such as a classroom:

> There is a necessity to constitute a special study of didactics that is both supported by psychology and distinct from it […]. Adapting to a classroom is really different than doing psychology with students of the same age. It is absolutely excluded that one can directly draw didactics lessons from psychology. (Piaget, in Morf, 1971, pp. 4-6, our translation)

Echoing Piaget's thoughts are those of Schoenfeld (1999), who argued that practically useful and theoretically enriching educational research should entail more than the traditional "research leads to development" perspective (see also Stokes, 1997). Instead, Schoenfeld proposed a model for educational research that places psychology and the research conducted in real-world settings (e.g., classrooms) in a dialectical arrangement. That is, researchers from both domains can work together so that the efforts of one informs the efforts of the other in an iterative, and even recursive, fashion, but at the same time, in such a way that can further the disciplinary objectives of both fields. In our specific case, we propose that the respective fields of educational psychology and *didactique des mathématiques* can be informed, and indeed enriched, by *teasing apart and maintaining explicit* the points of contrast in their respective questions, objectives, techniques, and interpretations.

Our experience confirmed that in collaboration, the work of educational psychologists and *didacticiens des mathématiques* cannot, and should not, be reductionist – that is, it should not be the objective to produce one single view on the study of students' mathematics. In this regard, we concur with Battista (2010), who delivered the compelling argument that the study of mathematics thinking and learning is best informed by research from a multitude of disciplines. A singular approach is unwelcome, as it would dampen the diversity of perspectives, thereby preventing opportunities for the fields to grow and enrich one another.

Conclusion

The collaboration we lived and outlined in this paper illustrates that it succeeded not by one "applying" the ideas of the other, nor by imposing on the other alternate ways of doing. Rather, its success can be conceived as an augmented sensitivity toward the need to agree and accept the other's perspective. In a very real sense, our interactions served to make the familiar unfamiliar; through our common goal of studying and supporting mathematical thinking, each of us had our ways of addressing that objective in ways that were to us routine, even comfortable. The collaboration pushed us out of our respective spheres of familiarity and forced us to recognize and confront our differences, rendering them explicit objects of discussion (and even of analysis for this paper!). From this emerged a critical examination of our own perspectives and sensitivities to those of the other. While we are not claiming that we produced a synergy that can serve as a model for all other collaborations between our two fields — a "how-to" guide for how our fields *should* collaborate — the experience worked *for us* in part because of our fundamental belief that learning from each other allowed us to more strongly justify, and at times question, our own assumptions and approaches. This appears to us as a central rationale for collaboration, whether at the individual or broader disciplinary level.

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Notes

i) In French, the term "*didacticien des mathématiques*" is used rather than "mathematics education researcher." Because JP is a francophone *didacticien des mathématiques*, we focus on this perspective in this paper. For obvious cultural reasons, however, differences do exist between mathematics education researchers and *didacticiens des mathématiques*. There is no place in this paper to elaborate on these, but more information can be gathered from the following edited volumes of Karp (2014) or Sierpinska and Kilpatrick (1998), as well as from the bulk of the interviews JP conducted in Proulx (2013a).

ii) HPO teaches in an elementary teacher education program housed in a child development unit at a large university in Canada. The program is not housed in a department of curriculum and instruction, as is traditionally the case in US institutions.

iii) Other examples of this sort, where *didacticiens des mathématiques* feel the urge and necessity to explore more complex views for the mathematics at stake are reported in Gauvrit (2012).

iv) Ways to maximize what is called the internal validity of a study have been covered extensively elsewhere (e.g., Campbell & Stanley, 1963).

v) Here is another disciplinary difference, where explanations for JP follow mathematical definitions, where "an explanation is the not the quest for underlying causes, but the *formation* of a string of phenomena, or its *modification*, or the *transformation* of this *modification*" (Cléro, 2015, p. 20, our translation).

vi) One additional discussion between HPO and JP that occurred concerned the nature of the verbal explanations as data and its reliability for gaining insight into students' thinking. Indeed, links between the verbal reports and the "actual" doings of a subject have been questioned for a long time in psychology (see e.g., Nisbett & De Camp Wilson, 1977). In that sense, one could say that JP's interest is in students' "verbal computations" and not in their "mental computations." Asking students to explain how they solve (orally and by coming to the board) was thus to collect strategies for solving tasks in that context.

vii) The expression "findings" is used instead of "results" to describe the *didactique des mathématiques* part, simply because *findings* is the usual expression, and it pays tribute to the importance and role of the researcher, the observer, as being "part of" what is said to have been *found* by him or her.

viii) Note that these strategies are not "new" in the sense that nothing similar has been attempted before in mathematics or even by Amy herself, but that they are generated *for* the task faced, locally, and thus reflect both the task and the individual's ways of solving (see Proulx, 2013b, on these matters of emergence and in relation to enactivism).

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Author Contributions

Both authors contributed equally to writing this paper and to the research reported.

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Appendix

The core defining feature of relational thinking depends on an understanding of the equal sign ("="; Freiman & Lee, 2004; Knuth et al., 2006). It has been well documented that many children see the equal sign as an indication to "do something," such as perform a calculation (e.g., McNeil & Alibali, 2005; Rittle-Johnson, Matthews, Taylor, & McEldoon, 2011; Sherman & Bisanz, 2009). Because this and related misconceptions are resistant to change (McNeil, 2014) — they persist through middle school (Knuth, Alibali, McNeil, Weinberg, & Stephens, 2005) and can even be activated at the university level (McNeil, Rittle-Johnson, Hattikudur, & Petersen, 2010) — it is important that teachers focus on the conceptual meaning of the equal sign as "the same as."

Even with a "sameness" conception (Jones, Inglis, Gilmore, & Dowens, 2012), however, students still engage in different types of reasoning about mathematical equations. Consider a student faced with the problem 25 + 17 = 22 + □. A student could solve it via computation by adding the amounts on the left side of the equal sign and then subtracting the amount on the right side (i.e., 25 + 17 – 22). Stephens (2006) called an approach based on calculation "computational thinking." Another way to solve the problem would be by thinking *relationally*. This would entail examining the relationship between the amounts on both sides of the equation, which could result in reasoning that "25 is 3 more than 22, so the answer must be 20 because to balance it out I would need a number 3 more than 17." In this specific example, then, relational thinking involves examining the equation, noticing the relationships between the 25 on the left side of the equation and the 22 on the right side, and compensating for the discrepancy by adjusting the 17 accordingly. The hallmark of relational thinking is that, while it can involve computation, it is not determined by computation. The focus is rather on mathematical structures and generalizations (Carpenter et al., 2005; Empson et al., 2011; Stephens & Ribeiro, 2012), which can add meaning to arithmetic and serve as an important foundation for algebraic reasoning. Indeed, Carpenter and his colleagues argued that a shift away from computation to meaning making marks an important transition to algebraic reasoning (see also Kaput, 1998).

Students who think relationally engage in transformations that are justified, often implicitly, by properties of whole number operations. When asked to think relationally about 99 × 3, for example, students can, and often do (e.g., Ambrose, Baek, & Carpenter, 2003), use the distributive property by transforming 99 into (100 - 1), so the product can then be computed by subtracting 3 from 300 (i.e., 99 × 3 = (100 - 1) × 3 = 300 – 3 = 297). Such transformations rely on the notion of *substituting* 99 for (100 - 1), a key element to understanding the equal sign. Jones et al. (2012) argued that alongside the notion of "sameness," the equal sign has a substitutive component, which implicitly justifies the mathematical transformations performed during relational thinking. Thus, each transformation results in an expression that is mathematically equivalent to the first: In this case, both (100 - 1) × 3 and 300 - 3 are equivalent to 99 × 3 and equivalent to each other. Another example (from Empson et al., 2011) illustrates the substitutive component quite clearly in the context of relational thinking with fractions. A child may think of ½ + ¾ = □ by substituting ½ + ¼ for ¾. This would make the computation more manageable because ½ + (½ + ¼) = (½ + ½) + ¼ = 1 ¼. In this way, relational thinking can lead to conceptual understanding of number and number properties, and may enhance students' thinking about mathematical generalizations (Cai & Knuth, 2011; Jacobs et al., 2007).

The transformations that occur in a mental computation are often achieved through implicit or explicit knowledge of number properties (Ambrose et al., 2003; Maclellan, 2001; Thompson, 2010). For example, faced with 113 - 30, a child could decide to transform 30 into 10 + 10 + 10 because removing 10 three times is more manageable than removing 30 at once. Maclellan (2001) argued that mental computation entails transforming expressions into others that look different *but are not changed in value* [our emphasis]. In this way, a child performing 113 - 30 mentally is essentially relying on key components of mathematical equivalence: sameness and substitution (Jones et al., 2012): 113-30 = 113 – (10 + 10 + 10) = 113 – 10 – 10 – 10. HPO thus entertained the idea that choosing a mental computation strategy is, in essence, an act of relational thinking because it relies on creating implicit transformations that rest squarely on equivalence. After choosing a suitable

mental strategy, a child must then execute the calculations mentally, which can occur in the context of relational thinking, but is not a defining feature.

Key terms and their definitions from this literature are presented in Table A1.

Table A1

Definitions of Key Theoretical Terms in Relational Thinking

Term	Definition
Canonical equation	An equation, traditionally used in elementary and middle-school mathematics classrooms, where there are no operations on the right side of the equal sign. Examples include: $a + b = c$; $a + b − c = d$.
Non-canonical equation	An equation, rarely used in elementary and middle-school mathematics classrooms (Carpenter, Franke, & Levi, 2003; McNeil, 2014; McNeil, Grandau, et al., 2006), that either have no operations (e.g., $a = a$), have operations only on the right side of the equation (e.g., $a = b + c$), or operations on both sides of the equation (e.g., $a + b = c + d$).
Open-number sentence	An equation with a missing number. Examples include __ $+ b = c + d$; $a = b +$ __; $a +$ __ $= c + d$.
True-false number sentence	Two expressions separated by an equal sign. The two expressions may or may not be equal, which would render the "equation" false. Examples include: $28 + 14 = 27 + 13$ (false); $(124 × 3) + 124 = 124 × 4$ (true) (Carpenter et al., 2003; Molina et al., 2008).
Operational view of equal sign	The misconception that the equal sign indicates a signal to "do something," such as "add all the numbers" and "the answer comes next" (Carpenter et al., 2003; Falkner, Levi, & Carpenter, 1999; McNeil, 2014; Rittle-Johnson et al., 2011; Seo & Ginsburg, 2003).
Relational view of the equal sign	The understanding that the equal sign represents an equivalence relation and means "the same as" (McNeil & Alibali, 2005; Sherman & Bisanz, 2009). Understanding the equal sign can also involve the notion of "substitution" (Jones et al., 2012).
Relational thinking	A form of reasoning that entails looking at numbers and expressions holistically and noticing relations among them (Carpenter et al., 2005; Empson et al., 2011; Jacobs et al., 2007).
Computational thinking	A focus on computation when analyzing open-number or true-false number sentences rather than on relations between the expressions on both sides of the equal sign (Stephens, 2006).