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Bridging Psychological and Educational Research on Rational Number Knowledge

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Abstract

In this paper we focus on the development of rational number knowledge and present three research programs that illustrate the possibility of bridging research between the fields of cognitive developmental psychology and mathematics education. The first is a research program theoretically grounded in the framework theory approach to conceptual change. This program focuses on the interference of prior natural number knowledge in the development of rational number learning. The other two are the research program by Moss and colleagues that uses Case's theory of cognitive development to develop and test a curriculum for learning fractions, and the research program by Siegler and colleagues, who attempt to formulate an integrated theory of numerical development. We will discuss the similarities and differences between these approaches as a means of identifying potential meeting points between psychological and educational research on numerical cognition and in an effort to bridge research between the two fields for the benefit of rational number instruction.

Keywords: number cognition, natural number bias, conceptual change, rational number development, educational implications

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Numerical cognition is a research area that appeals to mathematics education researchers, to cognitive-developmental psychologists and to neuroscientists. However, the researchers coming from these different fields approach numerical cognition in different ways in terms of theoretical perspectives, questions asked, methodologies used, and most importantly, of end goals (Berch, 2016). Thus there is great need for dialogue between psychological and educational research, particularly when it comes to implications for instruction. In recent years there have been several attempts to build bridges between the two fields by identifying potential meeting points. For example, Newcombe et al. (2009) discuss an explicit agenda for bridging research in psychology and mathematics and science education; and Alcock et al. (2016) make a point of highlighting the educational relevance of their research agenda for numerical cognition.

With this article we contribute to this effort, arguing that some of the research that lies in the intersection of cognitive-developmental psychology and mathematics education can be fruitful for both fields and very relevant for instruction. We focus on the development of rational number knowledge and we present our research program that is theoretically grounded in the framework theory approach to conceptual change (Vosniadou, Vamvakoussi, & Skopeliti, 2008; Vosniadou & Verschaffel, 2004). This research program focuses on the adverse effects of prior number knowledge in the transition from natural to rational numbers (aka the natural number bias) and the challenges it presents for students' numerical and algebraic reasoning. This phenomenon has been long noticed and studied by mathematics education researchers and has more recently attracted much attention by psychologists and neuroscientists. Apart from being a phenomenon of interest to both fields, research on the natural number bias has many important implications for instruction. We will compare our research program to two other programs that also illustrate the possibility of bridging research between psychology and mathematics education. The first is a program by Moss and colleagues who developed and tested a curriculum for fraction learning grounded in Case's theory of cognitive development (Kalchman, Moss, & Case, 2001; Moss, 2005; Moss & Case, 1999). The second is the research program by. Siegler and colleagues who attempt to formulate an integrated theory of numerical development (Siegler, 2016; Siegler, Thompson, & Schneider, 2011). Unlike the framework theory approach to conceptual change, these programs focus on similarities rather than on differences in the acquisition of natural and rational number knowledge. We will discuss these programs within the more general context of research on rational number learning and teaching. Finally, we will outline a research agenda that has the potential to bridge research between cognitive/ developmental psychology and mathematics education in order to better understand how rational number knowledge develops and how to improve instruction.

The Problem of Adverse Effects of Natural Number Knowledge on Rational Number Learning

Rational numbers are notoriously difficult for primary and secondary students and even for educated adults. Drawing on empirical evidence as well as on conceptual analyses coming from numerous studies, Moss (2005) summarizes several reasons that might underlie this difficulty: A complex knowledge network needs to be constructed that is based on multiplicative rather than on additive relations (e.g., Lamon, 2008); new symbols and representations are introduced that need to be understood and coordinated (e.g., Markovits & Sowder, 1991); early understandings of the unit and of the arithmetical operations need to be reconceptualised (e.g., Behr, Harel, Post, & Lesh, 1994; Fischbein, Deri, Nello, & Marino, 1985; Sophian, 2004). In addition, there are several conceptually distinct meanings, such as the part-whole aspect of fraction, fraction as quotient, as multiplicative operator, as ratio, and as measure attached to rational numbers that, again, need to be understood and coordinated (Behr, Lesh, Post, & Silver, 1983; Kieren, 1976). Thus, students face the challenge of mastering new material of highly complex content. Adding to the difficulty of this task is that prior knowledge and experience with numbers as natural numbers is not always supportive of rational number learning. In fact, the adverse effects of prior natural number knowledge on rational number learning is such a pervasive phenomenon that it has been attributed the status of a bias, and termed whole or natural number bias (Ni & Zhou, 2005; Vamvakoussi, Van Dooren, & Verschaffel, 2012).

The natural number bias has attracted the interest of researchers from diverse fields, such as mathematics education, educational psychology, cognitive and developmental psychology and neuroscience. It has been studied using different methodologies (i.e, paper-and-pencil tests, interviews, reaction time studies, brain imaging studies etc.), and different populations (i.e., spanning from infants to adults, and from novices to expert mathematicians). As could be expected, research on the natural number bias is multifaceted. One research strand documents systematic errors in incongruent rational number tasks (i.e., tasks that target aspects of rational numbers which are not compatible with natural number knowledge). Numerous misconceptions have been identified in various contexts, such as in the comparison of decimals (e.g., “longer decimals are bigger”, Resnick et al., 1989); in the comparison of fractions (e.g., “the numerical value of a fraction increases when the value of the terms increases”, Stafylidou & Vosniadou, 2004); and in arithmetical operations (e.g., “multiplication always makes bigger”, Fischbein, Deri, Nello, & Marino, 1985). The interrelations among students’ overreliance on natural number knowledge, their overconfidence in their erroneous answers, and the degree of resilience of their misconceptions have also been investigated within this research strand (Durkin & Rittle-Johnson, 2015; Fischbein, 1987; Merenluoto & Lehtinen, 2004).

Another strand of research looks into the reasoning processes that underlie the manifestation of the bias. Reaction time studies have been used to investigate the observation that natural numbers “come to mind first”, showing that—besides making more errors in incongruent tasks— people need more time to answer correctly in incongruent than congruent tasks (DeWolf & Vosniadou, 2015; Obersteiner, Van Dooren, Van Hoof, & Verschaffel, 2013; Vamvakoussi, Van Dooren, & Verschaffel, 2012; Van Hoof, Lijnen, Verschaffel, & Van Dooren, 2013). This line of research also investigates the possible connections between the manifestation of the bias and domain-general executive function skills such as inhibitory control (Gómez, Jiménez, Bobadilla, Reyes, & Dartnell, 2015; Vosniadou, 2014). In a similar vein, the interference of natural number representations in the processing of fraction magnitudes is investigated using methods based on the priming effect, on the Stroop effect, on the SNARC effect, and also with neuroimaging (e.g., Bonato, Fabbri, Umiltà, & Zorzi, 2007; Jacob, Vallentin, & Nieder, 2012; Kallai & Tzelgov, 2012; Meert, Grégoire, & Noël, 2009; Tzelgov, Ganor-Stern, Kallai, & Pinhas, 2013). Finally, there is a third strand of research in the intersection of psychology and neuroscience that touches upon fundamental questions about the roots of numerical cognition, particularly whether early representations of discrete quantity have primacy over representations of continuous quantity (see Ni & Zhou, 2005, for a thorough discussion of the issues involved).

Research related to the natural number bias provides valuable insights into the effects of the bias on rational number learning, the reasoning processes that underlie it, its connections to other cognitive functions, its neurological underpinnings and in a more general fashion, into numerical cognition. In this article we focus only on one aspect of this bias that we believe could be of value to mathematics instruction. Specifically, we discuss a research program that examines the natural number bias from the perspective of the framework theory approach to conceptual change, and we contrast it with two research programs that have clear roots in psychological theory but are also relevant to mathematics education research. Unlike the framework theory approach to conceptual change, which emphasizes differences between natural and rational numbers, the two other research programs emphasize similarities between natural and rational number knowledge. We will discuss these three approaches with a view to synthesizing ideas and suggestions for rational number instruction, taking into consideration relevant insights stemming from mathematics education research.

Emphasizing Similarities Between Natural and Rational Number Knowledge

The Moss and Case Rational Number Curriculum

Joan Moss in collaboration with Robie Case designed and evaluated a rational number curriculum grounded in Case's theory of cognitive development. This theory has been applied to the domain of mathematics accounting for the development of natural number concepts (Okamoto & Case, 1996), rational number concepts (Moss, 2005; Moss & Case, 1999), and function concepts (Kalchman, Moss, & Case, 2001). Moss and Case started with the premise that children's understanding of natural and rational number develops in similar ways: In both cases development starts with two distinct conceptual structures, the numerical schema (primarily digital, verbal, and sequential) and the global quantitative schema (primarily spatial, analogic, and non sequential). In the case of natural numbers, the first structure corresponds to the schema for verbal counting and the second structure corresponds to the schema for absolute quantity evaluation. In the case of rational numbers the first structure corresponds to the schema for splitting, in particular doubling, and the second structure corresponds to the schema for proportional evaluation. Development is described in terms of phases: In the first phase the two schemata are distinct. In the second phase they increase in complexity and are mapped onto each other. The coordination of the two schemata results in the emergence of a new "psychological unit" (more explicitly specified as a mental number line in Kalchman et al., 2001). In this phase children's understandings are limited to the simplest representatives of the field in question (e.g., small natural numbers, simple fractions such as 1/2 and 1/4, for natural and rational numbers, respectively). In the third phase the range of numbers is extended, and different representations are created. Finally, in the fourth phase the representations become more explicit, are co-ordinated, and the ability to move flexibly among them and to use them appropriately is developed.

Moss and Case (1999) argue that in addition to differences in the nature of initial conceptual structures, the developmental progression of natural and rational number understanding also differs in terms of the time when the two, initially distinct, schemas become integrated. By the age of 10, children appear to "have assembled a generalized understanding of the entire base-ten system and of the form of notation that is used for representing it" (p. 125). On the other hand, co-ordination of the initial conceptual structures pertaining to rational numbers—assumed to be in place by 9 to 10 years—starts at the age of 11 to 12, yielding "the first semiabstract understanding of relative proportion and simple fractions (especially 1/2 and 1/4)" (p. 125).

The rational number curriculum developed by Moss and Case was based on a number of well-elaborated ideas. First, they aimed at capitalizing on students' initial understandings and experiences. True to their analysis, they targeted 4th, 5th, and 6th graders; they started with the construct of ratio; they worked with situations rich with visual props (e.g., full, half full, and empty containers, number ribbons) building on children's assumed schema for proportional evaluation; and they presented ratio as percent, so that students could use their natural number strategies to deal with the numerical aspects of the tasks (e.g., halving, doubling). Then they introduced other symbolic representations for rational numbers (first decimal and then fraction notation) emphasizing the connection with the ones that were already familiar to students. Finally, they presented children with exercises in which all types of representations were to be used interchangeably, using number lines extensively.

This curriculum was evaluated in experimental, pre/post test interventions (Moss & Case, 1999; see also Moss, 2005). Moss and Case (1999) reported one such intervention with one experimental and one control group of 4th graders. The control group was introduced to fractions with the customarily used part-whole aspect of fraction. The experimental intervention consisted of twenty 40-minute instructional sessions spread over a period of 5 months. The results were very encouraging: The experimental group outperformed the control group in all tasks targeting conceptual understanding of rational numbers, and did not lack behind in tasks on procedural fluency. Moreover, the experimental group students showed qualitative differences in their reasoning about rational numbers, indicating that they engaged in multiplicative reasoning. On the contrary, the control group students still relied heavily on additive reasoning, resulting in errors.

The Siegler and Colleagues' Integrated Theory of Numerical Development

Similarly to Moss and Case (1999), Siegler and colleagues (Siegler, 2016; Siegler, Thompson, & Schneider, 2011) focus on similarities between natural and rational number concepts. Unlike Moss and Case, however, their emphasis is on fractions as measures, placing the understanding of magnitudes at the core of numerical development. In the first attempts to formulate their integrated theory, Siegler and colleagues (2011) proposed that numerical development is a process of:

> [...] progressively broadening the class of numbers that are understood to possess magnitudes and of learning the functions that connect that increasingly broad and varied set of numbers to their magnitudes. In other words, numerical development involves coming to understand that all real numbers have magnitudes that can be ordered and assigned specific locations on number lines (p. 274).

This progression is thought to reflect the gradual expansion of the mental number line to the right to include larger natural numbers, leftward to encompass negatives, and interstitially to include fractions and decimals (Siegler & Lortie-Forgues, 2014). The developmental process is described as including four overlapping trends (Siegler, 2016; Siegler & Lortie-Forgues, 2014) wherein: a) the size of (discrete) quantities is represented increasingly more precisely; b) connections are established between small symbolic (natural) numbers (0-10) and their non-symbolic referents (around 3-5 years of age); c) the range of numbers whose magnitude is understood extends to larger whole numbers (around 5-7 years for numbers to 100, and around 7-12 for numbers greater than 1.000); and d) the magnitudes of rational numbers are accurately represented (proper fractions from around 8 to adulthood; improper fractions as well as negative numbers from around 11 to adulthood).

The main tenet of the integrated theory of numerical development is that understanding the magnitude of numbers is instrumental for the development of numerical competence. This view is supported by many studies of natural as well as rational numbers (see Siegler, 2016 for a review). The major implication of the above for rational number instruction is to foster students' understanding of fraction magnitude. Supporting evidence comes from the evaluation of an experimental fraction curriculum that emphasized fraction magnitude—focusing primarily on representing, comparing, ordering, and placing fractions on a 0 to 1 number line—against a standard curriculum that emphasized the part-whole aspect of fraction (Siegler, Fuchs, Jordan, Gersten, & Ochsendorf, 2015). The experimental curriculum was implemented in three randomized control studies (Fuchs et al., 2013, 2014; see also Fuchs et al., 2016). The interventions consisted of 3 30-minute lessons per week spreading over 12 weeks, and targeted 4th graders (9-10 years old). The students were pre-and post-tested on several outcome measures of conceptual as well as procedural aspects of fraction knowledge. In all studies, the results showed that the experimental group outperformed the control group with respect to all outcome measures. In addition, the gap between high and low achievers decreased for the experimental group, whereas it increased for the control group. Siegler et al. (2015) highlighted the finding that improvement in understanding the measure aspect of fractions (but not improvement in understanding the part-whole aspect of fraction) mediated the effects of the intervention. This finding speaks to the importance of the measure aspect in rational number instruction.

The Framework Theory Approach to Conceptual Change

The framework theory approach to conceptual change (Vosniadou, 2014; Vosniadou, 1992, 1994; Vosniadou et al., 2008) was developed with the aim of describing and explaining the difficulties students face when they are exposed to counter-intuitive concepts in science and mathematics. It focuses on instruction-induced conceptual change as opposed to conceptual changes that occur spontaneously in development (Inagaki & Hatano, 2008). A key assumption of the framework theory approach to conceptual change is that from early on children organize their interpretations of common everyday experiences in the context of lay culture into few, relatively coherent, domain-specific framework theories. The rationale for applying the framework theory approach to conceptual change in the number domain was originally described in Vosniadou and Verschaffel (2004) and later further expanded in Vamvakoussi and Vosniadou (2010) as follows: In the domain of number, there is a great deal of evidence that children form a principled understanding of numbers as counting numbers, already at preschool age (Gelman, 2000; Smith, Solomon, & Carey, 2005). This initial understanding enables children to reason about natural numbers, to learn about their properties, and to build strategies in relation to natural number operations. Although there is evidence that young children are sensitive not only to discrete quantities but also to continuous quantities and their relations (Mix, Huttenlocher, & Levine, 2002), only the initial understanding of number as counting number is culturally supported in the early years. This is done via the use of linguistic tools (e.g., the number sequence), practices such as finger counting, and other activities embedded in parent-child play (Andres, Di Luca, & Pesenti, 2008; Carey, 2004; Greer, 2004). In addition, early mathematics education (typically from kindergarten up to grade 2) focuses on natural numbers, their properties and their operations, providing children with the opportunity to externalize and systematize their initial understandings of number as discrete quantities. Thus, before they are exposed to formal rational number instruction, students have already formed a rather coherent explanatory framework for number that is essentially based on features and properties of the natural numbers. So, for students numbers are associated with discrete quantity; they answer the question “how many?”; they are built on additive relations; and they obey the successor principle. Addition and subtraction are conceptualized in terms of counting, multiplication is conceptualized as repeated addition, and division is conceptualized as fair sharing, where the divisor is always smaller that the dividend. Each number is associated with one singular symbol, the unit is explicit and indivisible, there is a least positive number, and the size of a number can be judged by its position in the number sequence or by the number of its digits (see also Smith et al., 2005; Moss, 2005; Ni & Zhou, 2005; Vamvakoussi & Vosniadou, 2010). We stress that it is not assumed that students are aware of these assumptions; rather, these are implicit in nature (see also Fischbein et al., 1985, for a similar idea regarding the intuitive models of multiplication and division).

When non-natural numbers are introduced in the curriculum, although they are also called “numbers”, they clearly violate practically all the background assumptions underlying students’ initial framework theory of number. The question arises, how do students assign meaning to these new mathematical objects?

The framework theory approach to conceptual change is a constructivist approach. We argue that students draw heavily on their prior knowledge of natural numbers to make sense of non-natural numbers (e.g., reasoning by analogy). They typically employ additive mechanisms of learning to gradually enrich their initial framework theory of number with new information about non-natural numbers. When the incoming information, however, is not compatible with their knowledge base, the use of additive mechanisms destroys the coherence of the original structure and may result in fragmentation, internal inconsistencies, and the formation of misconceptions. From our theoretical perspective, this is a slow and gradual process during which a special type of misconception, namely synthetic conceptions, is formed. Synthetic conceptions reflect the assimilation of new information while retaining many of the background assumptions of the initial framework theory of number (see Vosniadou, 2013; and Vosniadou & Skopeliti, 2014 for a detailed discussion). Consider, for example, the “longer is bigger” error in the case of decimals, which is very common, particularly for younger students. This error becomes less common with age while the “shorter is bigger” error becomes more prominent and remains present even in adulthood (Desmet, Grégoire, & Mussolin, 2010; Stacey et al., 2001). This error is often interpreted as an intrusion of knowledge about fractions. From our point of view, this is a synthetic conception, reflecting the assimilation of information about non-natural numbers while retaining the idea that one can judge the size of a non-natural number by the number of its digits.

Applying the framework theory approach to conceptual change in mathematics (Verschaffel & Vosniadou, 2004; Vosniadou & Verschaffel, 2004) allowed us to generate hypotheses about the development of rational number knowledge, and also about students’ interpretations of literal symbols in the context of algebra. In the following we present the studies that tested these hypotheses, and interventional studies that addressed the expected difficulties of secondary students.

Psychological Experiments

Stafylidou and Vosniadou (2004) studied 5th to 10th graders’ understanding of the numerical value of fractions. They stressed that, unlike what students know about natural numbers, fractions cannot be ordered in terms of their position on the counting list, they are not bounded by a “smallest” fraction, and are not associated with a unitary symbol. In line with the framework theory approach to conceptual change, Stafylidou and Vosniadou predicted that students would make systematic errors related to these differences, which could be interpreted as synthetic conceptions of the fraction concept. Two hundred students from 5th to 10th grade were asked to a) write the smallest and biggest fraction that they could think of and explain their answers, and b) to compare and order fractions. The great majority of these students (89%) were placed in three categories corresponding to an initial and two intermediate states of fraction understanding. In the first category, the students did not take into consideration the multiplicative relation between the numerator and the denominator and considered fractions to consist of two independent natural numbers. In the second, fractions were considered to be always smaller than the unit, again ignoring the relation between the terms of the fraction. Only in the third category were students able to take into consideration this relation. Students in the intermediate categories exhibited synthetic conceptions of fractions. One such example comes from the majority of students in the third category who—

although apt to consider the relation between the terms of the fraction—still believed that fractions are bounded by a smallest and a biggest fraction.

In a series of studies, Vamvakoussi and Vosniadou (2004, 2007, 2010) investigated secondary students' (7th to 11th graders) understanding of the dense order of rational numbers. The distinction between discrete and dense order is a fundamental difference between natural and rational numbers: Within the natural numbers set, there is a finite number of intermediates between any two natural numbers. In other words, given a natural number, one can always find its successor. On the contrary, within the rational numbers set, there are always infinitely many numbers between any two numbers. In other words, the successor principle is no longer valid.

Vamvakoussi and Vosniadou (2004, 2007, 2010) found that students were very likely to say that there is a finite number (often, zero) of intermediates between two given rational numbers. This finding is consistent with the assumption that the successor principle is an important background assumption of students' framework theories of number. Other researchers have reported similar findings with students from primary to tertiary education (Giannakoulias, Souyoul, & Zachariades, 2007; Hannula, Pehkonen, Maijala, & Soro, 2006; Hartnett & Gelman, 1998; Merenluoto & Lehtinen, 2002). However, we were further able to establish that there are intermediate levels of understanding the density property. Students' judgments of the number of numbers in an interval were heavily affected by the kind of numbers used at the interval end points (i.e., whether they were natural or non-natural numbers, and by the representational form (fraction / decimal) of these numbers). More specifically, students were more prone to accept the infinity of intermediates between two natural numbers than between fractions or decimals. They were more likely to answer that there are infinitely many numbers between two decimals than between two fractions or vice versa. They were also more likely to say that these infinite intermediate numbers have the same representational form as the interval end points (i.e., infinitely many decimals between decimals, and infinitely many fractions between fractions). These results were replicated in a cross-cultural comparison with Flemish secondary students (Vamvakoussi, Christou, Mertens, & Van Dooren, 2011).

The above findings indicate that students' conceptualizations of rational numbers are far from the view of the rational numbers set as a unified system of numbers that are invariant under different symbolic representational forms. Rather, students seem to view (the positive) rational numbers as a collection of unrelated sets (natural numbers, decimals, fractions) that are allowed to behave differently with respect to order (discrete/dense). For the point of view of the framework theory approach this is a synthetic conception of the rational numbers set.

In a second line of research, Christou and Vosniadou (2005, 2009, 2012) investigated the effects of the natural number bias on students' interpretations of literal symbols in the context of algebra. Prior research has documented that when literal symbols are introduced in this context students are initially reluctant to accept that these symbols take their meaning in the domain of numbers—rather than being, for example, merely an abbreviation of an objects' name (e.g., "h" for "height") (Booth, 1984; MacGregor & Stacey, 1997). When students do start to associate literal symbols with numbers they typically believe that they stand for one single number, an "unknown" that needs to be discovered. Only later do they understand that literal symbols may stand for more than one number (Knuth, Alibali, McNeil, Weinberg, & Stephens, 2005). However, prior research in this area had not problematized the types of numbers that students tend to assign to literal symbols when they begin to realize that these symbols can take on multiple values.

Guided by the framework theory approach to conceptual change, Christou and Vosniadou (2005) hypothesized that students would initially show a strong tendency to substitute only natural numbers for variables. It was also predicted that even when students start to assign non-natural numbers, they would still be reluctant to accept any real number as substitute for variables. Instead, it was expected that there would be intermediate states or synthetic conceptions, where students would accept some, but not all, types of non-natural numbers.

A series of studies with secondary students (Christou & Vosniadou 2005, 2009; Christou, Vosniadou, & Vamvakoussi, 2007) using different methodologies, (such as open and forced-choice questionnaires, and interviews) supported this hypothesis, showing that students from 7th up to 10th grade tend to think that only natural numbers can be substituted for variables. For example, students answered that 2x stands only for multiples of 2, that a/b stands only for positive fractions, and that -b stands for negative integers. In agreement with our hypothesis, we also found that even the students who would accept non-natural numbers as substitutes for the variables, were not ready to accept any number as a possible substitute. In certain cases students accepted decimal numbers—but not fractions—as possible substitutes for variables, and in other cases they accepted fractions but not decimals. The finding that a literal symbol, denoting a (real) variable, stands only for specific types of numbers points to a synthetic conception of this mathematical notion. This finding has been replicated by a study with Flemish secondary students, who tended to assign only natural numbers far less frequently than their Greek peers but still had great difficulty to consistently assign any type of numbers to literal symbols (Van Dooren, Christou, & Vamvakoussi, 2010).

An interesting finding of the above research was that the students were particularly reluctant to accept that an algebraic expression that appeared to be negative (such as –b or -2x-1) could take on positive values, and vice versa (Christou & Vosniadou, 2009, 2012). We take this phenomenon to be yet another manifestation of the natural number bias. Consider that in the context of arithmetic the minus sign indeed denotes negative numbers. This can lead some students to decide on the spot that the phenomenal sign of an algebraic expression is its actual sign. Other students could reach the same conclusion by substituting the variable with natural numbers only. Indeed, an empirical study that focused directly on this issue provided evidence that students who were not willing to assign at least one non-natural number to variables were more prone to phenomenal sign errors in various mathematical contexts, such as inequalities and square root functions (Christou & Vosniadou, 2009).

To sum up, the framework theory has provided a new perspective on rational number learning and has generated novel predictions regarding the interference of natural number knowledge in the context of rational numbers as well as in the context of algebra. A series of empirical studies have confirmed these predictions and showed that there are intermediate states in the transition from initial to more sophisticated understandings of complex mathematical concepts, as students' initial ideas interact with the new counter-intuitive information coming from instruction forming synthetic conceptions. The insights into students' difficulties gained through this research were employed in the design of instructional interventions that addressed them.

Instructional Interventions

There is a number of principles for the design of instruction stemming from the conceptual change perspective on learning that are relevant for mathematics learning (Greer, 2004; Vosniadou, Ioannides, Dimitrakopoulou, & Papademetriou, 2001; Vosniadou & Vamvakoussi, 2006; Vosniadou et al., 2008). In the following, we illustrate how such principles were implemented in the design of experimental interventions with the aim to support secondary students' understanding of the dense order or rational numbers, and their interpretations of literal symbols in the context of algebra.

Vamvakoussi, Kargiotakis, Kollias, Mamalougos, and Vosniadou (2003, 2004) designed a sequence of 4 tasks requiring from students to use, evaluate, compare, and construct representations of numbers and the number line. These tasks were experimentally tested in two different settings, both of which encouraged students to express their ideas and to discuss and evaluate theirs and others' ideas and models. The experimental group (16 9^{th} graders) worked in Synergeia, a software designed to support collaborative knowledge building that provides a structured, web-based work space in which documents and ideas can be shared and discussions can be stored. The experimental group had online and offline access to their peers' answers and could write comments or respond to their peers' comments on their own answers. The control group (14 9^{th} graders) worked in pairs in their classroom, with paper and pencil, and the results were presented orally and then written on the blackboard by the researcher. One 45-minute session was devoted to each task. Both groups received the same pre- and post-test with tasks regarding the density of numbers. They were also interviewed after the intervention. The experimental group showed improved performance in the density tasks after the intervention and outperformed significantly the control group. In addition, the experimental students appeared to be more aware of the changes in their ideas about numbers before and after the intervention. It appears that the opportunity to express and exchange ideas using specific models, such as the number line, in an environment that allows for structured discussion was profitable for students.

Vamvakoussi and Vosniadou (2012) invested on the cross-domain mapping between numbers and geometrical objects, in particular the straight line, to foster students' understanding of the density of points and of numbers. Cross-domain mapping is considered a powerful mechanism for conceptual restructuring, because it allows for the transfer of inferences from a more familiar domain to the less familiar one (Gentner & Wolff, 2000; Vosniadou, 1989). The research on the density property presented above, and a pilot study reported in Vamvakoussi and Vosniadou (2012), had indicated that students were more prone to accept a) that there are infinite points on a line segment than infinite numbers in an interval, and b) that there are infinitely many intermediates (numbers or points) in an interval than that there is no successor at all. Vamvakoussi and Vosniadou used the line as a source domain and employed a *bridging analogy* (see Brown & Clement, 1989; Clement, 1993, 2008, for an elaboration on this instructional strategy) to decrease the gap between students' initial ideas and the sophisticated idea of points as a dense array. Specifically, the line was presented as an imaginary rubber band that never breaks, no matter how much it is stretched. We hypothesized that the "rubber line" analogy has the potential to help students grasp the idea of density, particularly the "no successor" principle, because it evokes students' experiences with a real world object (i.e., the rubber band). It is consistent with students' experiences with physical representations of the number line and it is associated, via the imaginary property of being unbreakable, with a recursive process, which is an easier way of accessing the notion of infinity for students. Finally, the number line analogy produces a sequence of segments rather than a sequence of points that can be deemed discrete.

We designed a text that provided explicit information about the infinity of numbers in an interval, made explicit reference to the numbers-to-points correspondence and used the "rubber line" bridging analogy to convey the idea that points (and numbers) can never be found one immediately after the other. The "rubber-line" text was experimentally tested against two other texts that contained the same explicit information. In addition it presented examples of intermediate numbers or figures illustrating the examples. Six classes of 8^{th} and 11^{th} graders (one experimental class per grade), 149 students in total participated in the study. They received a pre-test with density tasks in a arithmetical and a geometrical context, were administered the corresponding text, and then received a post-test containing all the pre-test tasks as well as 5 additional tasks that examined students' abilities to deal with the no-successor aspect of density. All groups profited from the explicit information about the infinity of numbers presented in the text. However, the experimental group (8^{th} as well as 11^{th} graders) outperformed the other groups in the "no successor" items of the post-test and was more consistent in providing correct answers and justifications for them.

In another experimental intervention, Christou (2012) exposed secondary students to a refutational lecture (Kowalski & Taylor, 2009) in order to address their difficulties with the phenomenal sign of algebraic expressions. This teaching strategy is similar to the use of refutational texts in instruction (e.g., Diakidoy, Kendeou, & Ioannides, 2003; Sinatra & Broughton, 2011; and Tippett, 2010 for a review). In such texts, and especially when the two-sided refutational argumentation methodology is used (Hynd, 2001), students' initial conceptions, beliefs, or ideas are directly stated and immediately refuted as a means of introducing a counter-intuitive concept or explanation (Dole, 2000; Hynd, 2001). Refutational texts not only provide arguments that falsify students' initial viewpoints, but also present the rationale of the new, to be adopted, perspective.

Christou's (2012) study had a pretest–posttest–retention test design with an experimental group who attended the refutational lecture, and a control group who did not (60 10^{th} graders in total). Both groups were pre- and post-tested with paper and pencil tests that used tasks regarding the sign of algebraic expressions in various mathematical contexts familiar to students (e.g., square root functions, algebraic inequalities, and absolute values). The refutational lecture used the main principles of direct instruction. More specifically, the students were given the definition of the real variable in algebra, and were explicitly told that literal symbols are used in algebra to stand for any real number, unless otherwise specified. Students' attention was called on to the differences between the (actual) sign of numbers in arithmetic, where the presence or absence of a minus sign indeed denotes a negative or positive value, and the (phenomenal) sign of algebraic expressions which does not necessarily denote an actual sign. Cognitive conflict was induced by providing examples and counter examples of natural and non-natural number substitutions to literal symbols in algebraic expressions, which either retained or changed their phenomenal sign. Students were also offered the strategy to always test the sign of an algebraic expression by substituting at least one negative number to the literal symbols. At the end of the intervention all the above points were discussed with the students. The whole intervention lasted about 25 minutes.

The results showed that the students who attended the refutational lecture made significantly fewer phenomenal sign errors compared to the students who did not attend the lecture. In addition, the learning profits were retained for at least one month after the intervention.

The intervention studies presented in this section addressed certain difficulties of secondary school students due to the natural number bias, by implementing principles for instruction and teaching strategies stemming from conceptual change research. These instructional interventions supported students to use, evaluate, compare, build, and discuss representations of mathematical constructs; promoted understanding via the use of analogies and bridging analogies; and explicitly addressed students' misconceptions using refutational arguments. The results showed that these teaching strategies can greatly foster the understanding of counter-intuitive mathematical ideas (see also Greer, 2004; Vosniadou & Vamvakoussi, 2006; Vosniadou et al., 2008).

The interventions discussed above targeted specific aspects of the natural number bias and were short-termed. For the problem of the adverse effects of natural number knowledge in rational number learning to be tackled more effectively, a long-term perspective on the planning and design of instruction is necessary. We argue that instruction should target the problem at the earliest possible time, before the discrepancy between natural and rational number knowledge and experience gets too large. In what follows we will ground this claim on a synthesis of psychological and educational research with reference to the research programs discussed above.

Synthesizing Psychological and Educational Research From the Point of View of the Framework Theory

The research programs presented above have similarities as well as differences in terms of their theoretical framing and the implications for rational number instruction. All research groups emphasize the importance of natural number knowledge in rational number learning. Moss and Case (1999) as well as Siegler and colleagues (Siegler, 2016; Siegler et al., 2011) highlight the similarities in the development of natural and rational knowledge. For Moss and Case this similarity lies in the form of the two processes whereas for Siegler and colleagues the similarity lies in the instrumental role of number magnitude. For the framework theory, rational number knowledge is built on natural number knowledge but requires its gradual modification and re-organisation, a process that involves considerable conceptual changes.

Moss and Case (1999) as well as Siegler and colleagues (Siegler, 2016; Siegler et al., 2011) agree that the development of natural number knowledge precedes the development of rational number knowledge. Moss and Case, however, argue in favour of building rational number knowledge on students' informal understandings of proportionality. In their proposal they emphasize the relational nature of rational numbers and consider natural number knowledge as a prerequisite for the development of the schema of splitting that allows for computations with percents. Siegler and colleagues, on the other hand, place greater importance on natural number knowledge. In fact, they argue than an important mechanism underlying the development of rational number knowledge is reasoning by analogy to natural numbers. Their basic claim for instruction is that "drawing the explicit analogy that fractions are like whole numbers in having magnitudes that can be ordered and represented on number lines may be helpful" (Siegler et al., 2011, p. 291). From our theoretical perspective, the account of rational number development offered by both research groups appears to downgrade the great discrepancy between natural and rational numbers with respect to early informal and formal experiences.

Finally, both Moss and Case (1999) and Siegler and colleagues (Siegler, 2016; Fuchs et al., 2013, 2014, 2016) consistent with their accounts of numerical development, target students that are older than 8 years of age. On the contrary, we believe that rational number instruction should begin earlier. As already stressed, natural numbers are culturally privileged by in and out of school experiences. It is of course very difficult to affect children's experiences in the first years of life. However, a window of opportunity to moderate the discrepancy between natural and non-natural knowledge is left unused if systematic attempts to teach rational numbers begin after the 3rd grade—as is true for the great majority of experimental intervention programs.

One overarching principle for instruction stemming from conceptual change perspectives on learning is that instruction should be designed not only on the basis of what is easy for children to understand at a given point in their cognitive development, but also by taking a long-term perspective, and by anticipating later expansions of the meaning of mathematical ideas and symbols as much as possible (Greer, 2004; Vosniadou & Vamvakoussi, 2006). Attempting to apply this principle to rational number knowledge development, one could ask: Why do students over-rely on natural number knowledge to make sense of rational numbers? We agree with Siegler et al. (2011) that a basic mechanism underlying rational number learning is reasoning by analogy. However, if the source domain of this analogy is limited to natural numbers, then the analogies drawn by students are more often than not unfavourable for further learning.

In a more general fashion, building rational numbers on the idea that are numbers that have magnitudes and can be placed on number lines, has certain limitations: It addresses the most abstract aspect of rational numbers that students will meet in their school career and it does not address the fundamental question of why rational numbers are considered numbers in a way that is meaningful for students. One should keep in mind that this is a question that has puzzled mathematicians for centuries and it entailed tremendous changes in the meaning of number (see Vamvakoussi & Vosniadou, 2012, for a discussion). Furthermore, it does not address (at least, not explicitly) the other meanings of fraction, notably fraction as ratio, and, most importantly, it does not address the problem of the adverse effects of natural number knowledge beyond the comparison of fraction magnitudes. This said, it should be noted that, despite the differences among the research programs presented here, the important role of the number line in rational number instruction is underlined by all three groups, a point that is also acknowledged by mathematics education researchers (e.g., Kilpatrick, Swafford, & Findell, 2001).

We value Moss and Case's (1999) idea that instruction should support students to build a conceptual background which can serve as a basis for rational number learning that does not depend heavily on natural number knowledge. Again we stress that, from the point of view of the framework theory approach to conceptual change, timing is an important factor to this end: Such a conceptual background should be in place before the firm establishment of a large discrepancy between natural and non-natural numbers.

At this point it is worth turning to the more general discussion on rational number teaching, focusing on suggestions for instruction that tackle, implicitly or explicitly, the problem of the natural number bias. First, it is important to note here that many researchers agree that overemphasis on the part-whole aspect when students first encounter fractions in instruction creates many problems in the long run. This is because the part-whole aspect of fraction, typically represented by the area model, actually evokes students' natural number knowledge and elicits additive rather than multiplicative reasoning (Moss, 2005; Ni & Zhou, 2005). Alternative approaches to the introduction of fractions include grounding instruction on ratio (e.g., Confrey, 1995); on fair sharing /equipartitioning (e.g. Confrey & Maloney, 2015; Nunes & Bryant, 1996; Streefland, 1991); and on measurement (e.g., Davydov & Tsvetkovich, 1991).

Second, it is also important to take into consideration that much effort has been put by mathematics education researchers in identifying deep similarities between natural and rational numbers, so that natural number knowledge can be used productively in rational number learning. To this end, a great deal of attention has been paid to the notion of the unit and to the operation of unitizing. These can serve as basic elements of reasoning with understanding in additive as well as in multiplicative situations and can be applied in the case of natural as well as rational numbers (e.g., Behr et al.,1994; Lamon, 1996; Sophian, 2004, 2008; Steffe & Olive, 2010). In particular, Steffe and Olive (2010) presented detailed evidence indicating that children are able to construct fraction knowledge through reorganization of their single-unit counting schemes, if placed in a carefully designed learning environment. Third, it has also been suggested that one needs to reconsider how natural numbers are taught in early math education, so as to facilitate the transition to rational numbers. In particular, Sophian (2004, 2007) pointed out that the role of unit in counting is largely neglected. She suggested that if more attention were paid to the importance of units in counting at the first years of instruction, the gap between natural and rational numbers could be decreased.

Based on the above, we argue that measurement is worth-investing on (see also Ni & Zhou, 2005; Schmittau, 2003). Indeed, measurement is at the heart of number concepts, and the reason why the introduction of non-natural numbers was necessary. Viewing counting as a special case of measuring can provide a meaningful explanation as to why non-natural numbers are members of the same category as the natural numbers. Measurement, of length in particular, fosters cross-domain mapping between continuous quantities and numbers, an important mechanism for numerical development (Smith et al., 2005). It is also a precursor of the cross-domain mapping between geometrical objects and number, which has been instrumental in the historical development of number concept, and has proved profitable for students (Vamvakoussi & Vosniadou, 2012). In fact, it lays the foundation for the much more abstract idea that non-natural numbers are numbers with magnitudes that can be placed on number lines, as Siegler and colleagues stress. Building rational number knowledge on measurement allows the introduction of the notions of ratio, equipartitioning, and also the part-whole aspect of fraction. Finally, measurement is intimately related to the notion of unit and the action of unitizing, which as discussed above, are instrumental for additive as well as multiplicative reasoning. Similarly to Sophian (2004, 2013), we believe that a plausible way to moderate the effects of the natural number bias in the long run is to invest on measurement in the early years curriculum. By highlighting the similarities between counting and measurement with emphasis on the role of the unit, natural and non-natural numbers can be introduced on the same background.

We are not the first to argue for the importance of starting rational number instruction earlier. Although this suggestion is not as popular as other approaches, there are still some researches that have taken this position (Ni & Zhou, 2005; Powell & Hunting, 2003). However, at this moment, there are several reasons why this suggestion could be more welcome. There is research-based evidence regarding children's early competencies regarding discrete as well as continuous quantity (Ni & Zhou, 2005; Mix et al., 2002; Sophian, 2004, 2007) and also on their competences regarding measurement (Sophian, 2007; Nunes & Bryant, 1996, 2009). In addition this suggestion comes at a time when great importance is placed on mathematics education in the early years (e.g., Moss, Bruce, & Bobis, 2016). Furthermore, the importance of the early development of multiplicative reasoning is acknowledged and reflected in the content of the early mathematics curricula worldwide. There are also voices arguing for introducing fractions earlier in the curriculum, even at kindergarten (Clements, 2004; Cwikla, 2014; Gupta & Wilkerson, 2015).

Finally, there is a strong trend in research on mathematics education to take a developmental approach to the design of curricula and instruction, which puts forward the notion of learning trajectories (Clements & Sarama, 2009; Sztajn, Confrey, Wilson, & Edgington, 2012). Learning trajectories are hypothetical, but empirically based and tested, models of students' transitions from less to more sophisticated understandings in a specific mathematical domain, under carefully designed instruction. Several learning trajectories have been articulated, for various content topics, including equipartitioning (Confrey & Maloney, 2015), and measurement (Barrett et al., 2012). The notion of learning trajectories help us to further concretize our suggestion: We argue that outlining a learning trajectory on cross domain-mapping between continuous quantities and numbers via measurement, particularly of length, extending to cross-domain mapping between numbers and the line, spreading from kindergarten to secondary school, could be a valuable tool to plan instruction on numbers on a long-term basis. This proposal builds on psychological as well as educational research and requires extensive research with different methodologies from both domains: content analysis, development of curricular material, small-scale experimental studies, longitudinal studies, and design studies. In this sense, it illustrates an example of bridging psychological and educational research, with a view to improve rational (and real) numbers instruction.

To summarize and conclude: In this paper we argued that some of the research that lies in the intersection of cognitive-developmental psychology and mathematics education can be fruitful for both fields and very relevant for instruction. We illustrated this point with reference to three research programs stemming from psychology that focus on rational number development and learning. We placed the discussion in the more general context of research on rational numbers, synthesizing findings and principles for instruction coming from educational and psychological research. Although we narrowed our synthesis to include research that tackles the issue of the adverse effects of prior knowledge on further learning, we acknowledge that this work is by no means exhaustive. Nevertheless, we hope to have contributed to the effort to establish more intense dialogue between the fields of psychology and mathematics education.

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