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Time as a Measure: Elementary Students Positioning the Hands of an Analog Clock

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Abstract

Elementary students have difficulty with the topic of time. The present study investigated students' actions to position hour and minute hands on an analog clock to indicate particular times of the day. Using one-on-one interviews with students in Grades 2 and 4 (n = 48), we analyzed whether students were more accurate for one hand indicator (hour or minute) versus the other as well as their solution approaches as they positioned each hand. We first present a quantitative analysis of student performance to document whether hour and minute hands posed differential challenges for students as they positioned hands to indicate particular times. Results indicate the hour hand is significantly more challenging to position accurately than the minute hand. Students' solutions reflected varied approaches, including consideration of the quantitative hour-minute multiplicative relationship, attention to part-whole relations, and matching numbers from the provided time to numerals on the clock. We discuss implications for theory and instruction, including the relationship of time to length measure learning trajectories and the current treatment of time in K-12 mathematics standards for the United States.

Keywords: clock, elementary math, measure, quantitative reasoning, time

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Elementary students have difficulty with the topic of time (Earnest, 2017; Kamii & Russell, 2012; Williams, 2012). Despite mathematics standards in the United States that suggest a mastery of key ideas related to time by Grade 3 (National Governors Association Center for Best Practices [NGA Center] & Council of Chief State School Officers [CCSSO], 2010), students in Grade 4 and beyond have documented challenges related to the multiplicative relationship of hours to minutes (Earnest, 2017; Kamii & Russell, 2012), quadratic functions with time as a parameter (Ellis, Özgür, Kulow, Williams, & Amidon, 2015), and variation with intensive quantities like speed that involve time (Lobato, Hohensee, Rhodehamel, & Diamond, 2012). In this study, we explore time understandings among elementary students related to standard units and in the context of an analog clock. We document students' approaches as they position hour and minute hands on the clock to indicate particular times. Given that the hands of an analog clock indicate different but related measurement units, in this paper we investigate if they treat the two indicators—i.e., hour and minute hands—differently and, if so, how we may understand their actions to position them. Because time is represented as geometric length on an analog clock

—similar to the treatment of time in other mathematical representations like timelines or function graphs—we seek to understand such actions as related to principles of measure.

Time is not just 'out there.' In fact, our capacity as a species to apply standard measurement practices to a quantity that is both invisible and untouchable marked feats of engineering, particularly the timekeeping inventions of the pendulum in the 17th century and the atomic clock in the 20th century (see Barnett, 1998; Cipolla, 1967/1978). Both discoveries reflected overcoming the same challenge: designing an instrument that treated units of duration as identical, invariant, and repeatable. Unlike other measurable attributes of the world, such as length or area, one cannot simply position a time tool at a starting point, with visual and tactile cues enabling the determination of a measure in some unit. Rather, humans needed to invent tools that replicated principles of measurement in other areas (e.g., length) with the measurement of time. Yet with the technological advances that led to the development of such tools that forever influenced day-to-day life in the modern world, present day students may now come to treat such tools in procedural ways that are distant from the original goal of measuring duration.

As with other measurement tools (e.g., a ruler or a protractor), the design of the analog clock accomplishes some of the underlying measurement principles for the user. In a typical clock design, equal intervals are arranged end to end and completely fill the perimeter of the circular clock face, thereby reifying duration as length units. The analog clock then represents multiple related units through the same twelve equal intervals to indicate the time of the day. We may consider the clock as featuring two or three number lines superimposed onto one another, each reflecting different units (hours, minutes, seconds). We may treat each unit as having a particular zero-point at 12 to represent midnight or noon (hour units), the beginning of the hour (minute units), or the beginning of a minute (second units), with units multiplicatively related (we note that any point may be treated as a zero-point; see Lehrer, Jaslow, & Curtis, 2003). Although an hour is sixty times greater in duration than a minute, the hour hand's pace on the clock is 1/12 the pace of the minute hand. With consecutive numerals positioned at equal intervals, one may coordinate the numeral and interval to interpret a specific unit.

In this paper, we investigate students' problem-solving actions when positioning hands on the analog clock to indicate particular times of the day. Given the complexities related to unit and interval on the clock, we are concerned in this paper with how students drew upon different units as they positioned the two hands. This study addresses the following two questions. First, do students position one hand indicator (hour hand or minute hand) more accurately than the other? If so, this would suggest that the task of positioning each of the hands is differentially challenging to students, and leads to the second question: How do students approach the task of positioning each hand to indicate a particular time? This question seeks to reveal the mathematical character across their actions, particularly as related to measurement.

We first consider psychological investigations related to duration followed by research in mathematics education pertaining to time in standard units. Tying these together, we then frame time in standard units and with standard tools as a part of students' quantitative reasoning and understanding of measure.

Psychological and Instructional Perspectives on Time as a Measure

To date, psychological investigations of students' attention to a temporal dimension have largely been separate from investigations of standard time units in mathematics education. Time understanding begins prior to 18-months of age as young children grapple with notions of sequence, such as past, present, or future (Harner, 1982; Piaget, 1954). As they develop so does their construction of temporal relations. Piaget's experiments, consistent with those involving number and geometry (Piaget, Inhelder, & Szeminska, 1960; Piaget & Szeminska, 1952), explored stages of development in the construction of temporal relations. Due to its invisible and untouchable character, temporal relations develop later than other logico-mathematical constructions (which refers to the mental relations individuals construct in their minds about the surrounding world; Piaget, 1951); we simply cannot see time as we can other quantities. As a result, researchers have found that individuals do not immediately disentangle the temporal dimension from observable phenomena, particularly distance and position (Fivush & Mandler, 1985; Piaget, 1969; Richards, 1982). For example, given two toy cars traveling for the same duration but different speeds (an intensive quantity that is a ratio of distance to time), a child may associate time traveled with the stopping position of cars, with the farther car interpreted as traveling for a greater duration based on distance or spatial information.

Studies have explored how early elementary-aged students begin to enumerate time. For time, informal measuring takes form through the application of two key principles: constancy and arbitrariness (Levin, 1989). Constancy refers to the same beat or physical motion recurring under the same conditions resulting in the same unit of duration, while arbitrariness refers to the irrelevance of the particular rhythm applied (though with a need for consistent tags or labels). Levin found that students made significant progress in working in particular with constancy in the second or third grade. According to Levin, arbitrariness develops after constancy and, for some students, may develop later than third grade. By Grade 3, students can apply a constant count to measure duration of short events, though they may still have difficulty recognizing the sameness of two equivalent durations when the counting rhythms (e.g., faster versus slower) resulted in two different enumerations. This study did not explore such principles in the context of standard units, and of course, the analog clock accomplishes constancy and rhythm for any user.

Such studies on the development of temporal relations, although valuable, typically did not involve the standard units or tools we use to measure time. In this area within mathematics education, research is scant, with a handful of studies in recent years that point to challenges students have in reading the analog clock, dealing with unusual groups of 12 and 60, and coordinating the multiplicatively related hour and minute units (Earnest, 2017; Earnest, Radtke, & Scott, 2017; Kamii & Russell, 2012; Williams, 2012). Findings have shown that elementary students in Grades 1-3 tend to have difficulty reading the hour hand on an analog clock when the minute hand is on the left side of the clock, for example, interpreting 2:50 as 3:50 due to the proximity of the hour hand to the "3" (Williams, 2012), a finding consistent with students' interpretations of number lines (Peled & Carraher, 2008). Williams found that early elementary students applied schemes for interpreting the analog clock—such as a proximity scheme in the aforementioned 2:50 example to interpret the hour hand based on the closest numeral—that reflected a discrete rather than continuous treatment of number. Another discrete treatment of clock properties involved students treating intervals on the clock as containers, with any point within that container treated as the same as any other point (Williams, 2012). Meanwhile, Kamii and Russell (2012) found that students across Grades 2-5 evidenced what they described as "an inability to coordinate hierarchical units," in particular because of challenges related to groups of 12 and groups of 60 (p. 309).

More recently, Earnest (2017) found that elementary students responded differently to problems about time depending on the analog or digital clock they used, in particular revealing how Grade 4 students treated hour and minute units separately. For example, a student interpreted an analog clock indicating 7:00 as "six o'clock." In this student's interpretation, the hour hand was positioned just before the 7 and, therefore, within the 6-7 interval. The student's explanation involved treating hour and minute indicators as independent from one another: the hour hand was located in the 6-7 interval thereby indicating 6 while the minute hand was on the 12 to signify o'clock. We might expect other individuals to similarly conclude the hour hand as being just before the 7 yet, based on the relative position of hour hand at the end of the 6-7 interval and the minute hand on the 12, interpret the time as 6:59 rather than 6:00. In that study, the different analog and digital clocks led to solution pathways that often reflected a separate rather than coordinated treatment of hour and minute units.

Current school practices related to early time development do not necessarily highlight the measurement aspects of time; in other words, time may be treated differently from its quantitative or measurement properties. In fact, time is often explicitly referenced as a read of the clock. For example, Common Core State Standards in Grade 1 state that students are expected to "tell and write time in hours and half-hours using analog and digital clocks" (NGA Center & CCSSO, 2010, p. 16). Following these and prior standards (National Council of Teachers of Mathematics [NCTM], 1989, 2000), elementary curricula have focused on the clock through discrete reading of isolated positions of time to the hour and half hour, or interpreting the hour hand first and then separately interpreting the minute hand (Kamii & Russell, 2012; Males & Earnest, 2015). There may be good reasons for such a pragmatic application of the tool, as intelligent behavior means that we may use a clock to measure time across the day without needing to attend to the quantity being measured. One may treat 6:00 pm, for example, in accordance with the routines that structure our lives related to making dinner, doing homework, or getting ready for a night shift of work. Such treatments are quite different than 6:00 pm as a measure of six hours elapsed from a noon origin.

Yet, as developmental psychologists have cautioned in the past, procedural treatments of tools and notation run the risk of students "learning procedural rules without the proper understandings implicit in [their] procedural uses" (Schliemann, 2002, p. 302). Time may be a part of routinized activity across a day, but the clock that indicates the time is a complex measuring tool featuring multiple related units. Prior calls have challenged an instructional focus on procedural uses of measurement tools (like rulers or protractors) without highlighting the quantitative referents underlying their procedural uses (Kamii & Clark, 1997; Moore, 2013; Stephan & Clements, 2003). Nonetheless, despite everyday applications of time, research on how students come to understand standard tools for and concepts related to time as a measure remains minimal.

Prior Knowledge to Support Time Understandings

To further situate the present analysis, we consider the knowledge that, based on related work in quantitative reasoning, we believe to be reasonable precursors to working with the clock and reasoning with time as a measure. We then consider the separate but related quantities of hour and minutes as units that change in relation to each other.

Skip Counting and Units Coordination — Students' competence with number typically develops prior to Grade 2 (Gelman & Gallistel, 1986; Saxe, Guberman, Gearhart, Massey, & Rogoff, 1987; Steffe, 1992). Curriculum and instruction build on students' understanding of number to support early time understandings in conjunction with clock reading, such as counting by 5s to identify the number of minutes past the hour. Such counting action builds on prior understandings related to iterating units and the construction of composite units. After entering first grade typically able to count by ones, students engage in skip counting by 2s, 5s, 10s, and other patterns. When individuals engage in such skip counting, they do so using *iterable composite units*

(Steffe, 1992), or units of units. In particular, students' prior experiences with iterable composite units supports treating numerals on the clock as groups of 5 minutes to position the minute hand.

For example, consider the time 4:30. In order to position the minute hand, one may skip count by 5s around the clock, thereby drawing upon prior work with iterable composite units to treat each numeral on the clock as a count of groups of 5. Yet this method is not the only one we may expect. Alternatively, one might instead (or additionally) consider the clock face as representing 60 minutes and partition the clock into halves with the minute hand one-half of the distance around the clock. Or, based on the common instructional focus in first and second grades on time to the hour and half hour (Males & Earnest, 2015), students may draw upon a procedure or memorized shortcut for the minute hand position for an x:30 time (e.g., that minute hand points down, or that the "6" on the clock stands for 30). Across these possibilities, individuals do not necessarily have to consider minute units as further partitionable in order to accurately position the minute hand indicator.

Accurate placement for the hour hand may draw upon different mathematical ideas as compared to the minute hand. Whereas the minute hand may draw upon composite units for the common procedure of counting by 5s, the hour hand is more likely to involve the partitioning of linear units (or, equi-partitioning; see Steffe, 2001). For example, for 4:30, one might first find the numeral "4" on the clock that matches the 4 in 4:30 (or, perhaps one might count by units around the clock starting at 12 until reaching 4). Yet one must also then consider the hour interval between 4 and 5, as the 30 minutes of 4:30 reflects a ratio of this quantity of minutes to the 60 minutes of an hour. One may use this ratio to position the hour hand one-half of the distance between four and five. In this way, the length-based representation of hour units on the analog clock draws upon partitioning and fractional understandings (see Lamon, 2007; Steffe, 2002, 2003; Tzur, 1999). Yet, one may additionally or alternatively rely on a rule independent of a fractional treatment (e.g., the hour hand for an x:30 time must be in the middle of two numbers), or even draw upon the numeral alone to position the hand directly (and inaccurately) on the 4.

Unlike the representational treatment of interval on an analog clock, the representation of digital time uses a value to indicate the hour independent of the amount of time elapsed past that hour (e.g., the 4 in 4:30). On an analog clock, the minute hand provides redundant information already provided by the hour hand, yet this is not the case with digital time. The digital clock's treatment of time is similar to the role of a quotient with a remainder in the U.S. division algorithm. In both cases, one may or may not consider the value (minutes or remainder) to be part of a whole. The 30 of 4:30 indicates 30/60, or ½, of an hour has elapsed from 4:00. This is similar for the expression 14 ÷ 4 with a solution "3 R2" (or, 3 remainder 2) for which the R2 represents a ratio of the dividend remaining to that divisor, or 2/4. Although we want all students to have conceptual understanding, successfully reading a clock or calculating a quotient with a remainder does not necessarily require conceptual understanding of part-whole relations and instead may rely on procedures.

Accurately positioning the hands involves reasoning across multiplicative units and, therefore, the action of units coordination (Steffe, 1992, 1994, 2001). As one engages in units coordination, Steffe (1992) considers it necessary to coordinate the two composite units—e.g., hours and minutes—"in such a way that one of the composite units is distributed over the elements of the other composite unit" (p. 264). Research has documented the role of units coordination in fractions concepts (see Hackenberg & Tillema, 2009; Izsák, Jacobson, de Araujo, & Orrill, 2012) as well as algebraic reasoning (e.g., Hackenberg, 2013). A time such as 4:30 is intended to represent two different but related units, with each of the different units multiplicatively related and with coordinated though separate origins. For the hour interval from 4 to 5, the 30 minutes are distributed across half of the hour interval. Although one may apply procedures to read a clock, an individual engaging in units coordination to interpret hour and minute hands is doing mathematical work important in various mathematical pursuits.

Quantitative Reasoning and Covariational Reasoning — Quantities refer to measurable attributes of a situation (see Thompson, 1994), with the process of quantification referring to how one assigns—or imagines assigning—numerical values to such measurable attributes (Thompson, 2011). As Thompson and others indicate (e.g., Moore, 2013; Moore & Carlson, 2012; Thompson & Carlson, 2017), values are a particular category of number that reference the result of an executed or imagined measurement process. In fact, when students do not attend to numbers as values in real world problems, their problem solving may become an "ungrounded debate" (Smith & Thompson, 2008, p. 110) about choosing numbers rather than reasoning about the relationships among quantities (Sowder, 1988).

Scholars have identified quantitative reasoning as supporting students' capacity to reason mathematically about properties of the world around them and as central for students to develop meaningful and connected mathematical understandings (Ellis, 2007; Hackenberg, 2010; Moore & Carlson, 2012; Thompson, 1994; Thompson & Carlson, 2017). For these researchers, a focus on quantitative reasoning can support students' conceptualizing, reasoning about, and operating on quantities and relationships in real-world situations and draws heavily on students' everyday experiences in the world. Thompson (2011) emphasized the need for quantities to become central to mathematics education, further stating that, "[t]oo often quantities, such as area and volume, are taken as obvious, and hence there is no attention given to student's construction of quantity through the dialectic object-attribute-quantification. Instead, textbook writers and teachers just use them to teach" (p. 34). As stated above, time is not just 'out there'; rather, it is a human invention resulting from efforts to quantify duration. Specifically related to time, Males and Earnest (2015) found that opportunities to learn time on the clock as featured in common elementary mathematics textbooks often reflected procedures related to clock reading, with fewer opportunities for conceptual understanding related to time as a measure of an attribute of the world.

An additional reason to investigate how students treat unit and intervals on the analog clock is related to the role of time in later grades. A treatment of time as a measure is critical to the construction of key intensive quantities in mathematics and science. While an extensive quantity references quantities that one can count or measure, an intensive quantity refers to those that are composed as a ratio (Schwartz, 1988). Thompson and Carlson (2017) stated that for one to evaluate intensive quantities—for example, speed—one must be able to differentiate both measures of distance and measures of time. They position time as a key quantity at the root of intensive quantities (see also, Keene, 2007). Lobato et al. (2012) found evidence of an eighth grade student's efforts to solve a problem involving speed; in their conclusions, they determined that the student's solution did not reflect attention to elapsed time, and that this in turn seemed to restrict that student's conception of speed. In other words, time is a key element of later mathematical pursuits, underscoring the importance in considering how students treat time as expressed in analog clock properties.

Students begin to consider quantities in the world through experience. In their research on students' developing theory of measure, Lehrer and colleagues have underscored that students' measurement understandings are a product of their own curiosity and everyday experiences (Lehrer, 2003; Lehrer et al., 2003). A child's own theory of measure develops from explorations of their worlds and, in doing so, gradually attending to the various attributes of objects, areas, or events. Such a framing for learning to measure contrasts sharply with instruction focusing on the procedural use of tools. As mentioned above, scholars have stressed that, to support generative and flexible understanding, students' learning must not be bound to the standard cultural tools used to measure those attributes, such as a ruler or a clock (Lehrer, 2003; Lehrer et al., 2003; Schliemann, 2002; Stephan & Clements, 2003). They cautioned against learning procedures for tool use independent of the quantity the tool was designed to measure. In other words, a focus on analog clock reading independent of duration does not support a conceptual foundation for time. Although an exclusive focus on tool use may solve local problems such as reading time on the clock for benchmark hour and half hour times or, as Moore (2013) revealed, using a protractor to find the answer to an angle measure problem, students do not necessarily develop generative or flexible understandings together with the mastery of such procedures.

In addition to considerations related to time as a quantity, the clock features at least two hands to indicate two separate but related quantities. Research investigating students' problem solving with time and elapsed time has drawn attention to challenges related to the relationship of hours and minutes (Earnest, 2017; Earnest et al., 2017; Kamii & Russell, 2012; Williams, 2012). One may coordinate particular placements of the two hands, such as the hands described above for 4:30: if the minute hand is halfway around the clock, the hour hand must be at the midpoint of an hour interval. Yet one may also attend to the movement of one hand and its relative movement with the other. We consider units of time—specifically, hours and minutes—as quantities with magnitudes that vary simultaneously. In other words, the two quantities *co-vary*: a clock's measure of time reflects movement in the hour hand and a proportional, corresponding movement in the minute hand.

Given the relationship of hour and minute hands, we consider such a relationship as an early example of students encountering quantities that co-vary, though encountering hours and minutes does not in and of itself mean that an individual is engaging in covariational reasoning. Carlson, Jacobs, Coe, Larsen, and Hsu (2002) define co-variational reasoning as "the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (p. 354). Covariational reasoning is critical in the development of mathematical understandings, as it refers to the modeling of relationships between quantities in an applied context (Carlson, 1998). More recently, Thompson and Carlson (2017) described covariation as epistemologically necessary for individuals to develop conceptions of functions and, due to the critical role of functions in mathematics, as fundamental to students' mathematical development. In the analysis below, we consider the role of covariation as students position the two hands of the clock, with such considerations reflecting our own descriptions of the situation based on students' actions rather than definitive claims about students' mental actions.

We reflect once again on warnings from developmental psychologists regarding students learning procedures for a tool without the underlying conceptual meaning. Consider a study focusing on precalculus students' angle measure understandings. Moore (2013) argued that instruction that treats angle measure as a set of procedures for calculating or reading a protractor "fails to address the quantitative structure behind the *process* of determining an angle's measure" (p. 227, emphasis in original). Thus, the rich web of mathematical connections that led to the use of these calculations may be absent. Moore contends that introducing angle measures without a focus on quantification "likely sends students the message: Use these numbers to perform calculations and find other numbers, but do not worry about what the numbers, arcs, and calculations mean" (p. 244). In fact, the dominant cultural tool for angle measure, the protractor, may support a successful procedural read to determine the correct answer without treating the resulting number as a quantity. Instead, the answer is merely a number on the protractor found through a procedure. Considering Moore's (2013) findings in relation to time on an analog clock, individuals may construct or interpret the clock based on procedures of clock reading rather than as a measure of or as a relationship between quantities. Rather than an individual treating numerals on a clock as reflecting values, one may treat them as numbers absent of any attribute of the real world. Although we see importance in successfully reading a clock (or, "telling time"), we liken this skill to Moore's procedural description of using a protractor.

To begin to investigate how students draw upon features of the analog clock, we explore in this study students' actions as they position each hand. In our study, we investigate first whether students face differential challenges when positioning hour and minute hands and, if so, how we may understand their problem-solving efforts. Given that the analog clock measures time over a day, we would expect that approaches leading to correct hand positioning might reflect ideas of measure and the relationship of hour and minute units; conversely, we would expect that approaches leading to incorrect hand positioning might more likely reflect treatments of number identification and repeatable units without necessarily coordinating hour and minute units.

Method

Participants

Participants included students in Grades 2 (n = 24) and 4 (n = 24) from six elementary schools in urban, suburban, and rural areas in western Massachusetts. All schools were classified as Title I schools, meaning that all have been identified as having a high percentage of students from low-income families, with 51% of all state public schools categorized as such. Interviews were conducted between January and April, 2015. Further demographic information is reported in Earnest (2017).

As reported in Earnest (2017), Grade 2 students were selected because standards indicate these students have previously mastered time to the hour and half hour in Grade 1 and are currently working on time at the 5 minutes (NGA Center & CCSSO, 2010). Their responses thereby offer a window onto student thinking for students that, according to current practice, are considered to be at an appropriate age for time instruction. Grade 4 students were selected because, per standards, time as reflected on clocks has been mastered in prior grades. Because of this, their performances would illuminate how students that have completed direct instruction related to the focal content solve the same set of problems. Interviews with teachers of participating Grade 2 classrooms indicated they worked sporadically on time during short morning activities or routines. None of the Grade 4 teachers focused instruction on time in this academic year.

To identify a range of participants for the present study, the sample was drawn from a larger study involving students in the six elementary schools (see Earnest, 2015b, 2017). An assessment designed for the larger project featured questions related to time and elapsed time. We administered this assessment in the six schools to 612 students across Grades 2-5. Focusing on Grades 2 and 4 participants, we used the total assessment score to assign 144 students (72 in each grade) to one of three interview groups, with a goal of having each interview group reflect a range of learners from high to low within and across participating classrooms. Students were assigned to one of three interview conditions to form groups of 24 students per grade in each condition. The 48 students in the present paper include 24 students in each Grade 2 and 4

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assigned to the condition featuring an analog clock with independent hands (for an analysis comparing performance across the three conditions featuring different tools, see Earnest, 2017).

Interview Procedures

Students were interviewed individually, and all interviews were videotaped. Each interview took approximately 30 minutes. We report here the tasks analyzed for this paper.

Materials and Tasks

Interviews included a focus on students' activity as they positioned hands on the analog clock as well as their descriptions of their own problem solving. Students were provided a common classroom manipulative: an analog clock with hands that are independent from one another. As reported in Earnest (2017), such a tool has affordances and constraints. Any user arguably must be intentional in positioning each of the two hands, thereby providing a window onto their thinking about exactly where each hand should point. The independent hands further enable our analysis of students' actions as related to each of the hands. At the same time, one may position the independent hands in any configuration, meaning that one may choose a configuration that does not reflect an actual time in our timekeeping system.

Interviews featured seven questions focusing on the positioning of hands on the clock (Appendix includes all tasks featured in interviews). Hand Positioning tasks were designed based on both prior literature (Kamii & Russell, 2012; Williams, 2012) and 1.5 years of piloting in elementary schools. We asked students to position hands for seven different times that included: time to the hour (7:00), time to the half hour (4:30), time on the first half of the clock (10:10), time on the second half of the clock (2:50), and a time with a minute value less than 10 (9:03). In addition to this, there were two relative time tasks (half past 11, quarter past 8) phrased in common U.S. vernacular that referenced hour units only.

Procedures

To present the Hand Positioning tasks to students, the interviewer asked the student to show a time on the clock (for example, "Show me what 2:50 looks like on this clock"). When asking the question, the interviewer turned over a card that featured the question in written language, including digital notation or the words for relative time (Figure 1). After the student positioned the hands to show the target time, the interviewer then asked that student how that student solved it. When appropriate, the interviewer would ask additional questions to clarify the student's explanation.

Show 7:00 on the clock.	Show 4:30 on the clock.	Show 10:10 on the clock.	Show 2:50 on the clock.	Show 9:03 on the clock.	Show *half past 11* on the clock.	Show *quarter past 8* on the clock

Figure 1. Playing cards for administering the Hand Positioning tasks.

To address the potential of an order effect of tasks, three orders of interview tasks were presented across participants. All items were featured in each order. A principle for ordering tasks included that Hand Positioning tasks should not follow one another. As reported in Earnest (2017), there were no performance differences based on the order in which tasks were presented (p = .341).

Analytic Methods

To evaluate performance on Hand Positioning tasks, we coded first the position of hour and minute hand indicators through triangulating observable hand position in video with students' actions and explanations. Four coders independently analyzed video and transcripts to determine the position of hands; such evaluations were double coded. First, we used video data to identify a precise position for both the hour and minute hand as a measure between 0 and 12; for example, an accurate position for 4:30 would indicate an hour hand positioned at 4.5 and the minute hand positioned at 6.0. Of course, measurement involves error, and precision with a relatively small clock—from the perspective of the user and the coder—is challenging. We coded any position within 0.1 of the accurate placement on either side as accurate (e.g., 4.4-4.6 for the hour hand for 4:30). To address error on the part of both participants and coders, we also triangulated visual measures in video of hand positioning with students' explanations to inform the value we assigned. For example, someone positioning the hour hand for 9:03 (or a coder viewing video of a child doing so) may do so in a way that doesn't well distinguish whether the hand is exactly on the 9 (to indicate 9:00) or just slightly above it (to indicate a time just after the top of the hour). During interviews, the interviewer asked for clarification for the exact position. For such cases, we consulted transcripts and video to triangulate with our observed measure (for example, a student that explained "It's not on the 9 but it's really, really close to it" would be coded differently (9.2) from "It's directly on the 9" (9.0)), thereby adjusting coding for issues related to precision and fine motor skills. Any discrepancies in coding were resolved in weekly team meetings by consulting data and reaching consensus. We use the measures resulting from this coding in our results to evaluate students' accuracy in positioning the hands.

We also coded for observed actions students employed as they positioned each hand. We open coded transcripts and video using the constant comparative method (Corbin & Strauss, 2008) to develop and refine a coding scheme. Using these data sources, we triangulated student explanations together with gestures and tool use as evidence for a solution approach. The first round of coding began with our research team creating a codebook, applying working codes to video, and, when applicable, proposing amendments to coding definitions or the creation of a new code. For this round, we remained agnostic about which hand a code might describe, and in fact attempted to define codes such that they applied to either hour or minute hand indicators. We followed this process through 15 of 48 cases. In the second round of coding, we applied codes to the 48 total interviews. Four research team members each coded 24 students' solutions (12 from each grade), with all cases double coded. In cases of coding discrepancies, coding decisions were discussed until reaching consensus, with elaboration of coding definitions provided when applicable. Due to the scant research in the area of time to date, the interview protocol featured a total of 23 questions (see also, Earnest, 2017) and, given the young age of participants, we limited follow-up questions during interviews to those needed for clarification purposes. Our codes reflect students' initial or revised approach to positioning the hands with follow-up prompts intended to clarify their intent rather than investigate the multiple ways they may think about the problem. For this reason, we do not see the boundaries of categories presented below as sharp but rather as evidence of observed action and explanation. Although we assigned students one code as described below, we acknowledge the possibility that an individual student may have considered more than one of the approaches described below simultaneously without necessarily verbalizing it. We further elaborate on this below when considering limitations to the present study and the need for future research.

Results

The results are presented in two sections corresponding to each of the research questions. First, we examine performances during interviews on the Hand Positioning tasks for both Grades 2 and 4. We then present a qualitative analysis of students' strategies applied to position each hand for the set of tasks.

Performances on Hand Positioning Tasks

To evaluate students' performances with hour and minute hands on the clock, we compared accuracy for each indicator (hour hand indicator and minute hand indicator) for each of the seven tasks. Figure 2 displays success rates for each hand indicator for the two grades (means and standard deviations are provided in Table 1). A Two (Hand Indicator) × Two (Grade) repeated measures analysis of variance (ANOVA) revealed a main effect for Hand Indicator, $F(1, 46) = 70.469$, $p < .001$, with better performance with the minute hand as compared to the hour hand. A main effect also emerged for grade, $F(1, 46) = 11.131$, $p = .002$, with Grade 4 students outperforming Grade 2 students.

Table 1

Means and Standard Deviations for Each Grade for Correct Performance on Hand Positioning Tasks

	Grade 2		Grade 4	
Hand	***M***	***SD***	***M***	***SD***
Hour Hand	2.88	1.83	4.54	2.24
Minute Hand	5.04	1.78	6.42	1.10

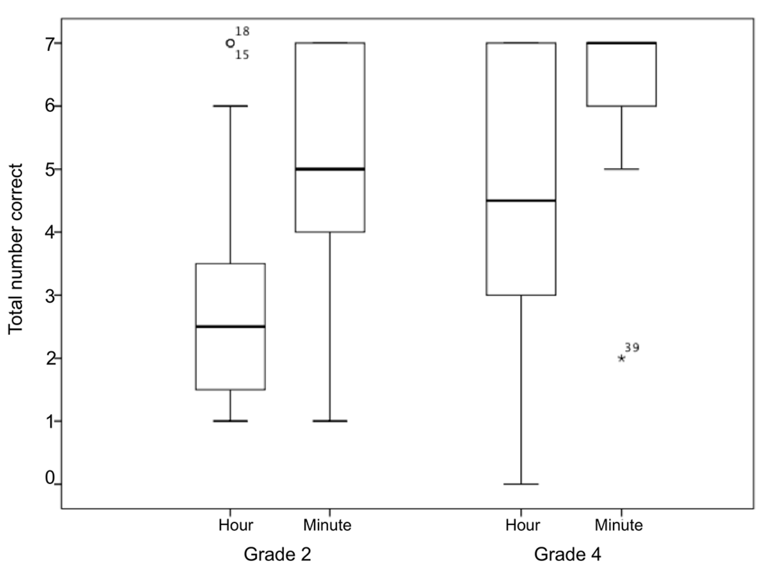

Figure 2. Boxplots showing performance on the seven Hand Positioning tasks for both hour and minute hands for each grade.

There was no significant Hand Indicator × Grade interaction, indicating that the discrepancy in performance across grades for both hour and minute hand placement was not significant (p = .548). Grade 2 students were significantly more successful for the minute hand than hour hand (t(23) = 6.397, p < .001), and Grade 4 students likewise were significantly more successful with minute hand placement compared to hour hand placement (t(23) = 5.480, p < .001); effect sizes for both groups from minute hand to hour hand placement (d = 1.1797 and d = 1.065) were found to exceed Cohen's (1988) convention for a large effect (d = .80). Regardless of grade, there was a significant decrease in accuracy from minute hand to hour hand, indicating a greater challenge in accurately positioning the hour hand.

Students' Solution Approaches to Position Hour and Minute Hands

Given this difference in performance, we turn to our second research question regarding students' actions to position each hand, with particular attention to the mathematical character across actions. Figure 3 displays success rates on the seven tasks for both hour and minute hands, and it reveals that time to the hour (i.e., 7:00) was met with overwhelming success for both hands. Among other times also presented in digital notation, students consistently performed better with the minute hand as compared to hour. For relative times (i.e., half past 11, quarter past 8), hour and minute hand accuracy showed less variation.

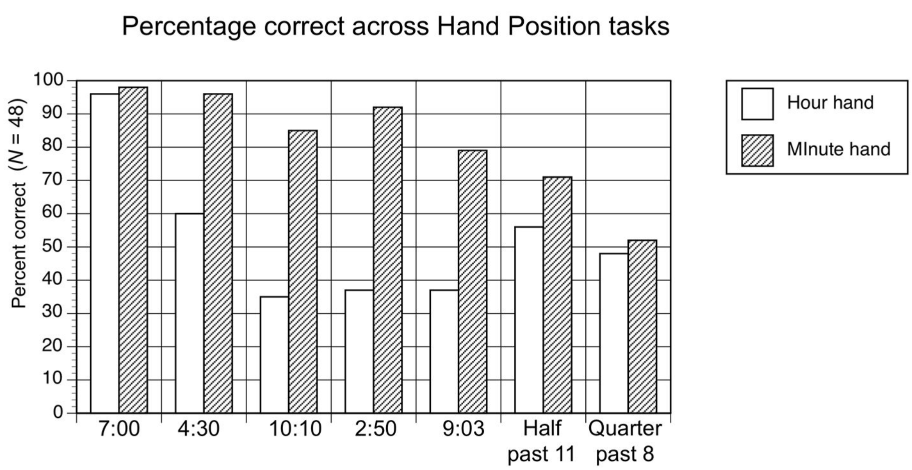

Figure 3. Performance across Hand Position tasks for both hour and minute hands.

Students' Solution Approaches

In this section, we provide an overview of codes that emerged in our analysis for the seven Hand Positioning tasks. We first present the five codes emerging from analysis of video and transcript. After this overview, we present our analysis of students' problem-solving approaches for each hand across all Hand Positioning tasks in both grades.

Our analysis of the 48 students' solution approaches resulted in five codes (Table 2): Container, Number Matching, Unit Iteration, Qualitative Coordination, and Quantitative Coordination, with idiosyncratic or unclear strategies coded Other. Table 2 features code names with examples as well as the frequencies across the 672 possible instances (2 hands on each of the 7 problems for 48 students) and, given all instances for just that code, the percent of correct responses. First, 27 responses were coded as Container. Consistent with Williams (2012), students treated a particular interval as a discrete container, with any point in that container the same as any other point. Second, Number Matching refers to an approach in which students matched the number from the time in the prompt with a number on the clock. At times, this included the application of a fact that remained unexplained in the interview (e.g., “I know 6 is 30”). Third, Unit Iteration refers to when students iterated a unit or composite unit, e.g., counting by 5s around the clock. To do so, students identified some zero-point on the clock—typically (though not always) the 12—and applied that counting pattern in order to position the hand. Fourth, we coded some approaches as Qualitative Coordination; to position one hand, students provided a descriptive relation with the other hand. Fifth, solution approaches reflected a Quantitative Coordination in which, to position one hand, they quantified its relationship with the other hand. A final Other category was applied for idiosyncratic responses or in instances that were unclear or for which the interviewer did not sufficiently ask questions to clarify the student’s explanation.

Table 2

Results of Coding for Students’ Solution Approaches

Approach	Number of Instances	% of total (*N* = 672)	% correct	Example
Container	27	4.0%	29.6%	*Hour hand for 4:30*. “It’s 4, so the hour hand is in the 4-space.”
Number Matching	218	32.4%	51.8%	*Hour hand for 4:30*. “The hour hand points at 4.” *Minute hand for 4:30*. “30 minutes is on the 6.”
Unit Iteration	124	18.5%	89.5%	*Minute hand for 4:30*. “I started here [at 12] and counted by 5s until I got to 30.”
Qualitative Coordination	99	14.7%	75.8%	*Hour hand for 2:50*. “I put that almost at 3 because it’s almost three o’clock. <*gesturing to the minute hand positioned at 10*>”
Quantitative Coordination	101	15.0%	91.1%	*Hour hand for half past 11*. “I did it [the minute hand] halfway, and then I did [the hour hand] halfway to 12.”
Other	103	15.3	52.4%	Idiosyncratic responses or interviewer did not ask for clarification.
Total	672			

Table 2 indicates that the most common solution approach for hand positioning was Number Matching, meaning that students matched the number for the hour in digital or relative time or applied an unexplained rule for the minute hand (e.g., “30 minutes is on the 6”) involving a numeral on the clock. Despite its prevalence, this approach was successful only about half the time. Two codes were highly successful among students: Unit Iteration (89.5%) and Quantitative Coordination (91.1%), with Qualitative Coordination (75.8%) just behind this. Regardless of overall success for a particular solution approach, all codes emerging from data at times led to correct hand positioning and at other times to incorrect hand positioning.

Recall that quantitative analyses indicate differential performances across the two hands, with a better performance with the minute hand as compared to the hour hand. To further address our second research question, we now turn to an analysis of solution approach to understand students’ actions as related to each unit indicator.

Problem-Solving Actions for Hour and Minute Hands

Figure 4 features data for each of the five solution codes plus Other for both hour and minute hand for each grade. Each bar of the bar graph reflects correct and incorrect hand placement for the 168 possible instances (7 tasks × 24 students in each grade). Solution codes were not evenly distributed across both hands, and in fact, particular approaches (i.e., Container and Unit Iteration) were almost exclusively applied to only one of the two hands.

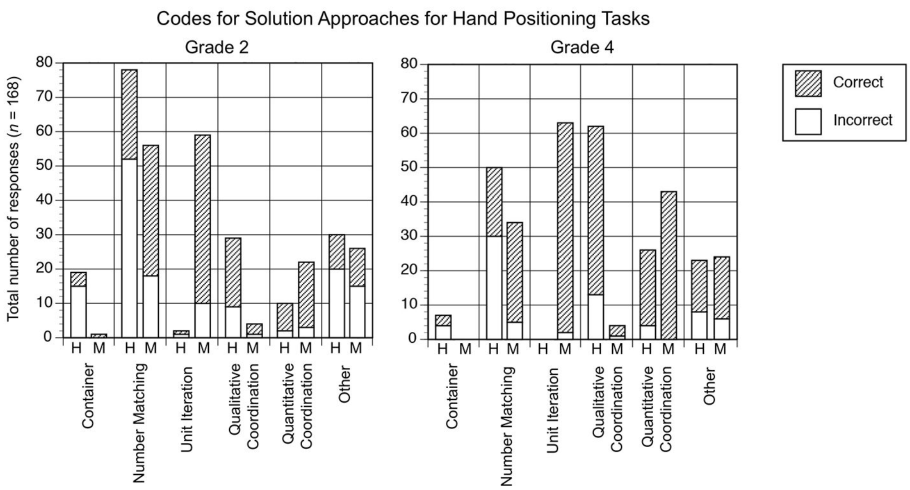

Figure 4. Codes for hour and minute hands.

We first consider the code Container, which was applied to just 4.0% of the 672 possible instances. The application of this code was almost exclusively for positioning the hour hand (only once was this code applied to the minute hand), with Grade 2 students (29 instances) applying it more frequently than those in Grade 4 (7 instances). In this approach, students treated the interval as a container rather than as continuous. Consistent with container interpretations while clock reading (Williams, 2012), students treated any position within the interval as the same as any other position. For example, for the time 9:03, one second grader inaccurately positioned the hour hand at 9.4 almost midway between 9 and 10 (Figure 5a) and explained, "I just put it anywhere between this box <*indicating the interval between 9 and 10*>." One second grade participant was coded as Container for the hour hand for 4:30, yet instead of treating an interval as a container, she treated the printed character ("4") as a container. She positioned the hour hand at what appeared to be just after the 4 between 4 and 5, but then explained her positioning while gesturing to the symbol "4" on the clock (Figure 5b): "Like right on that little—if you were at 4, it's going to be like right there," indicating the numeral "4" printed on the clock.

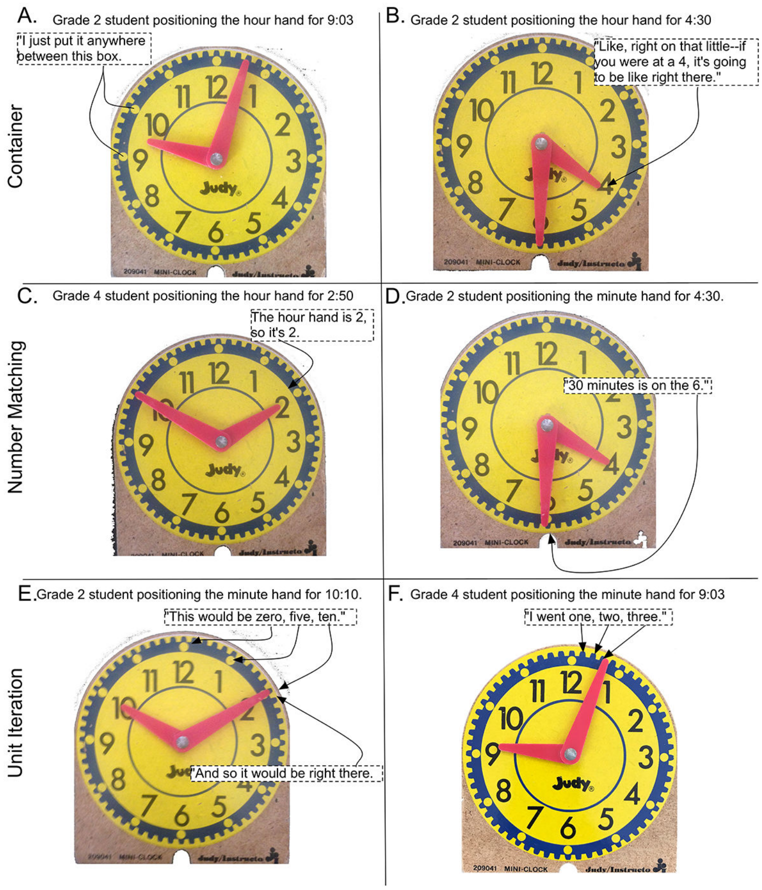

Figure 5. Students' strategies to position hour and minute hands included (a-b) Container, (c-d) Number Matching, and (e-f) Origin.

Number matching was the most common approach emerging from data, with 218 total instances (32.4% of all codes). With a success rate of 51.8%, students in both grades sometimes positioned hands accurately when number matching and sometimes inaccurately. Students in both grades were more likely to number match with the hour hand, though this code was still well reflected for minute hand positioning. When applied to the hour hand, students identified the hour value in the time provided with a specific numeral on the clock, positioning that hand at that numeral. For example, one fourth grade student indicating 2:50 positioned the hour hand (inaccurately) at the 2 and explained (Figure 5c), "The hour hand is 2, so it's 2." Meanwhile, a second grade student for 4:30 positioned the minute hand accurately on the 6 and explained (Figure 5d), "30 minutes is on the six." Note that, for such cases, there was no additional elaboration to indicate that student iterated units from a zero point (Unit Iteration) or considered the position of 30 as halfway between the 0 and 60 minutes of an hour (Quantitative), either of which would have led to a different code. For the time 7:00 specifically, we note an unusual yet expected trend of the same approach applied both hour and minute hand: The majority of students employed a Number Matching approach for both hour and minute hands in Grades 2 (19 of 24 and 15 of 24, respectively) and 4 (19 of 24 and 13 of 24), with all but one application leading to accurate placement. In other words, more than half of the Number Match approaches leading to a correct position were for time to the hour, suggesting that this particular approach may be insufficient for times that do not fall on the hour.

Unit Iteration reflected efforts to count from a zero point on the clock and was applied 124 of 672 possible codes (18.5%). Unlike the prior two codes, students were very successful when applying a Unit Iteration approach, with 89.5% of all instances reflecting an accurate position of that hand. Yet like the Container code, which students applied to the hour hand, Unit Iteration applied almost exclusively to positioning the minute hand. Typically, this approach featured students treating the 12 as a zero point, then counting by 5s until reaching the target multiple of 5, or by ones in the case of the time 9:03. For example, one Grade 2 student solving the problem 10:10 accurately positioned the minute hand at 2.0 (Figure 5e), explaining: “This would be zero, five, then, and so it would be right there.” A student in Grade 4 indicating 9:03 accurately positioned the minute hand at 0.6 counting the minute notches and stating (Figure 5f), “I went one, two, three.”

Qualitative Coordination was applied in 99 instances, which reflected 14.7% of all codes. When students employed a Qualitative Coordination approach, it led to accurate hand placement 75.8% of the time. A key aspect to this approach is that the placement of the target hand was described in coordination with the position of the other hand. Unlike the prior three codes, responses coded as Qualitative Coordination reflected efforts to treat the two hands as related to one another, suggesting potentially initial observations of how the two hands move together. Qualitative Coordination was almost always applied to the hour hand (29 times in Grade 2 and 62 times in Grade 4) as compared to the minute hand (4 times in both Grades 2 and 4). For example, a second grader showing 2:50 positioned the hand accurately at 2.8 (Figure 6a) and, gesturing to both hand indicators on the clock, explained, “I put that almost at the three, because it’s almost three o’clock.” A fourth grade student inaccurately placed the hour hand at 2.5 to show 2:50 yet, despite the inaccurate placement, appealed to the relationship of the two hands (Figure 6b), explaining, “[The minute hand] is heading to three o’clock. And the hour hand is moving from 2, and it’s moving to 3.” Despite attention to a relationship between the two hands, this student inaccurately positioned the hour hand.

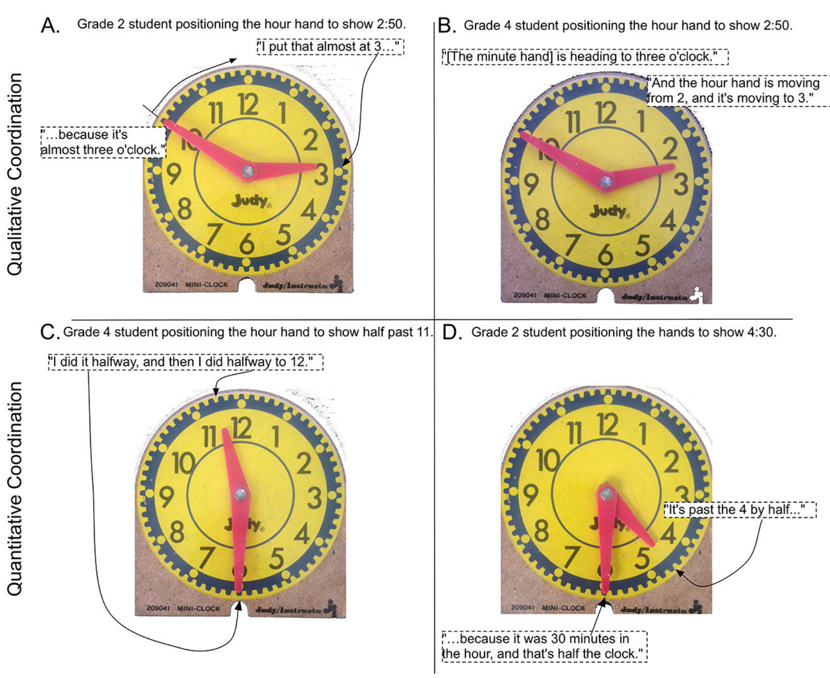

Figure 6. Students’ strategies to position hour and minute hands included: (a-b) Qualitative Coordination, and (c-d) Quantitative Coordination.

Finally, a Quantitative Coordination code was applied 101 times, about the same frequency as the Qualitative code. The accuracy rate for this approach, 91.1%, was the highest of all codes emerging from data; students applying a Quantitative Coordination approach for either hand were likely to reach an accurate placement. As with the prior qualitative coordination code, we consider a quantitative coordination solution approach to reflect students' attention to the relationship of the two units. For example, one fourth grader for half past 11 positioned the hour hand at 11.5 and appealed to the position of both hands together to explain: "I did it [the minute hand] halfway, and then I did [the hour hand] halfway to 12 <*Indicating interval from 11 to 12*>." When Quantitative Coordination was applied to the minute hand, participants similarly explained the minutes in reference to the hour. For example, a second grader was coded as Quantitative Coordination for both minute and hour hands, explaining an accurate placement for both hands at the same time: "It's past the 4 by half because it was 30 minutes in the hour, and that's half the clock." Of the 36 total applications for the hour hand across both grades of a Quantitative Coordination approach, 29 of these were for either 4:30 or half past 11. The non-standard groupings in 12 and 60 for hours and minutes may render some hour positions (e.g., 2:50) more challenging to quantify and then partition the hour interval than others (e.g., 4:30), a conjecture that came through in data.

Our analysis of solution approaches reveals that, beyond the statistical difference in accuracy, the very approaches students employed to indicate the time on an analog clock differed for hour and minute hands. Recall that students had more success accurately positioning the minute hand as compared to the hour hand. Overall, we found that minute hand strategies favored Unit Iteration (counting from a zero point), an approach seldom applied to the hour hand among our participants. For the hour hand, we found that students were most likely to Number Match, an approach that was accurate for time to the hour yet highly inaccurate when applied to any time not on the hour.

Discussion

The goal of the present study was to reveal the problem-solving approaches students apply when positioning hands on an analog clock. As we found, students were more accurate with the minute hand as compared to the hour hand, and their approaches to position each hand varied. In this discussion, we first consider the results of the solution approach analysis for each of the hands. We then consider implications of the study, in particular as related to efforts in both mathematics education and developmental psychology. We further consider instructional implications for how to support students' understanding of time as expressed in hand positions on the analog clock.

Hand Positioning Tasks and Problem-Solving Approaches

Students' competence with number and counting typically develops prior to Grade 2 (see Gelman & Gallistel, 1986; Olive, 2001; Saxe et al., 1987; Steffe, 1992). In this study, these competencies emerged in students' actions to position hands on the analog clock. The most common overall approach to position the hour hand, also well represented among minute hand approaches, was Number Matching, in which students targeted one of the two numbers (e.g., 2 or 50 for the time 2:50) and matched it to a number on the clock. Along with the code Container, the Number Matching approach seems to reflect a student's discrete treatment of numbers on the clock. Particularly among second graders, this discrete rather than continuous treatment of number has likely dominated their instructional experiences thus far. Yet, time is not a discrete quantity. Thus, approaches reflecting discrete treatments of number often led to inaccurate positions, with Number Matching having an overall accuracy rate of 51.8% and Container 29.6%. For Number Matching, more than half of those accurate positions reflected the time 7:00, suggesting that such an approach reflects a procedure successful only for time to the hour.

The Unit Iteration code was a highly successful approach, with almost 90% of all applications leading to correct hand placement. To employ this approach, students drew upon their prior work with iterable composite units, specifically counting by 5s around the clock. We observed this approach almost exclusively for the placement of the minute hand, for which the 12 was treated as the zero point. In doing so, students counted by 5s around the clock using each numeral to keep track of groups of 5 until landing on the target multiple of 5 (or counting three minute notches in the case of 9:03). Importantly, this approach reflects a treatment of the tool as having a zero point for minutes, a key principle for measure (Barrett et al., 2012; Lehrer et al., 2003). However, only one in 48 students iterated units to position the hour hand. While students considered 12 as a zero point for minutes, results lead us to wonder if students conceptualize the hour without a zero point. Such an approach would be a reasonable practice for efficient use of a clock, though at the same time may point to a challenge in teaching and learning about time as a measure. Furthermore, we note that this common approach to position the minute hand, although overwhelmingly successful, did not overtly consider the relationship of hour and minute units.

Quantitative and qualitative codes, though not the most represented approaches for either hour or minute, were quite successful (91.1% and 75.8% accuracy, respectively). In this approach, students described one unit in coordination with the position of the other unit either descriptively (Qualitative) or numerically (Quantitative). We note two important aspects to the two coordination approaches. First, for the hour hand, students partitioned an hour interval to position the hour hand in relation to what we describe as the ratio of minutes past the hour to one hour (of course, students in our study did not describe this as a ratio). This approach reflects a treatment of interval as partition-able (Lehrer et al., 2003), a key concept of measure. Researchers have identified such partitioning (or, equi-partitioning) as an important component of fractional understanding (Boyce & Norton, 2017; Hackenberg, 2010; Norton & Wilkins, 2012; Steffe & Olive, 2010). Our results indicate that the capacity to equi-partition may support an accurate placement and corresponding mathematical reasoning about the position of the hour hand.

Furthermore, each of the strategies, even when applied incorrectly, demonstrated students' efforts to treat the two units as related. We conjecture that such coordination may indicate partial or incipient consideration of hours and minutes as units that co-vary. As mentioned above, covariational reasoning is a cognitive activity involving coordinating two varying quantities that change in relation to each other (Carlson et al., 2002), and is fundamental to developing conceptions of functions. Although we do not conclude that these students had achieved covariational reasoning or where on a possible trajectory they may be, we see important aspects to qualitative and quantitative descriptions as two quantities that change in relation to one another.

Implications

Theoretical Implications

Although studies in temporal relations and those in standard time units in mathematics education have been conducted separately, we see promise in the convergence of these two areas for both theories of mathematics learning and the design of instruction. Research in developmental psychology has established that the capacity to attend to time as a property of the world comes later than other logico-mathematical constructions (Fivush & Mandler, 1985; Piaget, 1969; Richards, 1982). Yet, with concerns related to psychological constructions of a temporal dimension, researchers conducted such studies absent of standard notation and tools for measuring time. Meanwhile, research in mathematics education related to time has remained scant.

Given findings above related to qualitative and quantitative coordination codes in particular, a theoretical implication of this study is that understanding time on the clock builds on students' quantitative reasoning related to units coordination (Steffe, 1992, 1994, 2001) along with partitioning related to constructing fractional units (Boyce & Norton, 2017; Hackenberg, 2010; Norton & Wilkins, 2012; Steffe & Olive, 2010). In our data, students' approaches and explanations that treated hours and minutes as related quantities involved multiple mathematical considerations related to time units involving number recognition, partitioning, and the equivalence and joint movement of two separate hand indicators. Although such efforts were not always accurate, we consider even these inaccurate approaches as potentially along a pathway towards a robust understanding of time as a measure represented on the clock. We see promise in further exploring connections between unit and interval on an analog clock with similar representations of time on number lines and function graphs (see Earnest, 2015a).

Quantities of hours and minutes are continuously changing values, a co-varying relationship that prior research has found to be particularly challenging even among pre-calculus and calculus students (e.g., Carlson et al., 2002; Moore & Carlson, 2012; Thompson, 1994). Moore and Carlson further made the case that students need increased opportunities to work with quantities that vary in tandem. Consistent with our contention that we ought to move away from clock reading as the endgoal of time instruction, we see the relationship of time units as potentially fruitful for exploring important mathematical ideas related to covariation. More research is needed in this area to investigate how time understanding may serve as part of the foundation for quantitative reasoning, covariational reasoning, and reasoning with intensive quantities.

We further consider a potential meeting point of developmental psychology and mathematical education in the developing body of research in learning trajectories (Clements, 2007) and specifically to elaborating a hypothetical learning trajectory (Simon, 1995) for time. Research has focused in recent years on integrating psycho-cognitive descriptions of students' thinking with pedagogical treatments of content in instruction. Research in the area of length measure learning trajectories has identified key moments we might reasonably expect among typical students in instruction (Barrett et al., 2012; Sarama & Clements, 2009; Sarama, Clements, Barrett, Van Dine, & McDonel, 2011; Szilágyi, Clements, & Sarama, 2013). For example, Steffe (1991) identified the conceptual ruler that we might expect as early as Grade 3 (about Age 8), which refers to one's ability to mentally iterate units even when not available along a particular object being measured.

Such research identifies milestones along the length measure learning trajectory and, given a need for further research related to time, provides critical information for researchers to consider a hypothetical learning trajectory (Simon, 1995) for time. Barrett et al. (2012) identified Grades 2 and 3 as a key transition when students become capable of iterating units, while research in temporal relations identified a similar point when students attend to constancy of time units (Levin, 1989). We imagine that the capacity for grappling with time in standard units—an invisible and untouchable quantity for which logico-mathematical constructions develop later than length—comes *after* these key transitions, and in fact this is corroborated in research independent of standard time units (Kamii & Russell, 2010). We imagine a hypothetical learning trajectory in which we build on length measure and further consider the development of quantitative and covariational reasoning with fractional understandings related to partitioning. Although further research is necessary, we contend that the placement of time instruction on an analog clock in Grades 1-3 is too early in the K-12 sequence. If students in early elementary grades are not ready to learn ideas related to time, time is unlikely to serve as a resource as they progress through mathematical and scientific ideas in upper elementary grades and beyond.

Instructional Implications

One does not need to consider time to be a measure when reading a clock or following a schedule across the day, as a time (e.g., 6:00pm) is associated with daily contexts that are likely more powerful. Despite this, we again draw upon Schliemann's (2002) warning regarding the procedural use of tools hiding the mathematical measures underlying their procedural uses. In the case of a clock, while we may read the hour without regard to where within an hour interval the hand is (the minute hand provides this information for us), such a treatment does not position time as a measure. Above, we considered Moore's (2013) thoughts on angle measure: one may successfully use a protractor to determine an answer for angle measure without an understanding of that number as a quantity. We share this concern in regards to time.

If we treat time as an intrinsic property of a clock rather than as a quantity the clock measures, students will likely come to understand the clock as telling the time and, conversely, the time as numbers the clock indicates. Yet, if we place quantification at the forefront of instruction for time, students may come to treat time on the clock as a measure of duration across a day, thereby enabling the symbolization of this quantity to serve as a foundation for other mathematical and scientific pursuits. As research with middle and high school students has indicated (Ellis et al., 2015; Lobato et al., 2012), non-quantitative treatments of time interfered with students' mathematical work with quadratic functions involving two co-varying quantities as well as intensive quantities. Reframing this, we see potential in quantitative approaches to time instruction as supporting—not interfering with—students' efforts in higher mathematics for which time is a common parameter.

Limitations

We identify three limitations of the present study. First, the present study sought to document how students positioned hands on the clock with a focus on understanding patterns across their approaches. Our interview protocol involved a series of 23 questions and, with the population of second and fourth graders, we asked clarifying questions only when necessary to understand that student's approach in order to keep interviews at an appropriate length for this population. A limitation to this data collection procedure is that students solving a problem in a particular way may have multiple ways of thinking about time rather than just a single approach to positioning the hand. For example, a student coded as Unit Interval to position the minute hand as 4:30 may in reality be drawing upon prior quantitative knowledge of fractions (30 minutes being half of the clock) in addition to counting by 5s. Furthermore, it is also possible that questions featuring AM and PM designations may have further supported students' responses in time across the day such as recalling what the clock looks like when starting to eat dinner at 6:00 p.m. Subsequent research ought to provide an in-depth analysis of student thinking to further investigate procedural limitations to the present study.

Second, the study presented above featured tasks using a clock manipulative that, although it enabled insight into how students treat clock properties, did not actually function as a clock itself. Though minimal, the studies involving time in standard units in mathematics education are typically independent of duration, the property of the world that time measures. Although a premise of our work is that duration and measure are critical to support time understandings on and off the clock, the present study did not actively engage students in measuring duration. We must therefore interpret our results humbly. We encourage future research to consider how duration itself may inform students' responses in interviews or instruction related to standard units.

Third, the present study did not consider how culture and language is related to time. Just as in the United States we have different ways of naming the same time (e.g., quarter past 8, 15 past 8, 8:15), other languages of the world have varied ways of referencing and even conceptualizing time (see Burny, Valcke, Desoete, & Van Luit, 2013; Lakoff & Núñez, 2000). Such considerations were not a part of the present study design.

Future Research

Despite the ubiquity of time in our everyday lives and as a parameter in mathematics, science, and engineering, there is a paucity of time research in relation to standard tools and notation, to psychological constructions of duration, and the relationship between the two. The present study seeks to provide research in this area, yet represents beginning steps. In future research, we hope investigations will further illuminate the relationship of number development and quantitative reasoning with understanding of time on the clock. Further, as mentioned in the limitations, the present study was conducted independent of duration, and we see a need for further research to investigate standard units of time together with opportunities to measure duration. If students have a robust conceptual understanding of time, and because time is a critical aspect of many scientific investigations, they will have key resources to engage meaningfully in important mathematics and science content. Future research is necessary to prove such an assertion.

More research is necessary to determine the appropriate place for standard time units to be explored in the K-12 sequence. Wherever its appropriate position, we see students' understanding of time and the relationship of time units continuing beyond fourth grade. As Smith and Thompson (2008) indicate, building quantitative reasoning is not a one or two-year program, but rather it requires focus across elementary and middle school years. Given its invisible character, time is a topic that students may come to understand across years beyond Grade 3. We see promise in considering time in relation to the years-long work of developing quantitative and covariational reasoning.

Research on early time development has much to benefit from coordinated treatments between psychology and mathematics education. In both disciplines, time in standard units has remained an under researched area. Our study points to students' own treatments of standard units for time measure on the analog clock and suggests that such treatments may be anchored in procedural uses of the clock rather than measurement aspects of time. If we expect students to draw upon time to investigate phenomena in the world around them, instruction must move beyond clock reading procedures and elapsed time calculations. Instead, we encourage research and practice to position time as a measurable property of the world, with cultural tools such as clocks reflecting key measurement ideas related to unit and scale.

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Competing Interests

The authors have declared that no competing interests exist.

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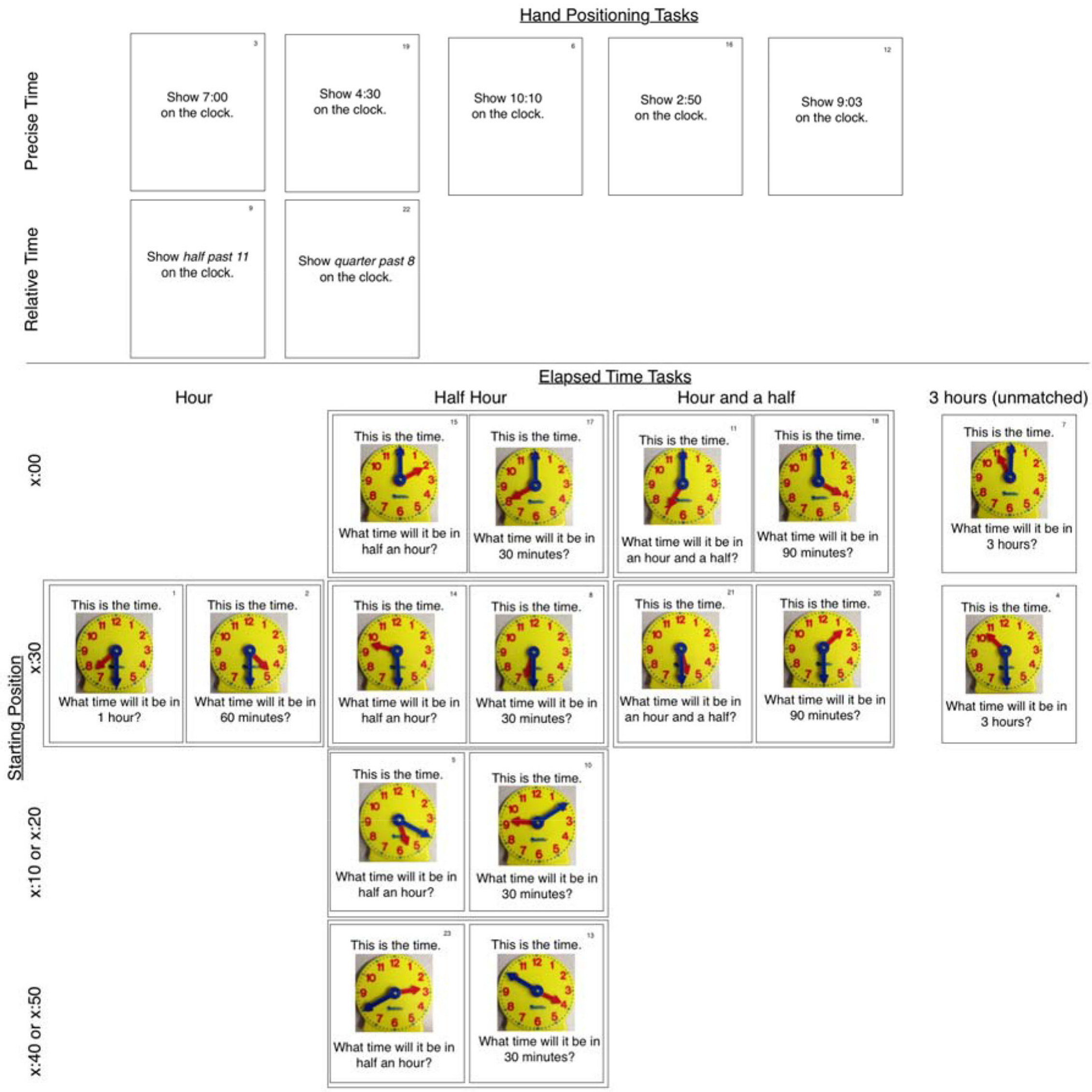
Appendix

Problems (as reflected on cards used during interviews), including seven Hand Positioning tasks and 16 elapsed time tasks (adapted from Earnest, 2017).



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