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How the Eyes Add Fractions: Adult Eye Movement Patterns During Fraction Addition Problems

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Abstract

Recent studies have tracked eye movements to assess the cognitive processes involved in fraction comparison. This study advances that work by assessing eye movements during the more complex task of fraction addition. Adults mentally solved fraction addition problems that were presented on a computer screen. The study included four types of problems. The two fractions in each problem had either like denominators (e.g., 3/7 + 2/7), or unlike denominators exhibiting one of the following relationships: one denominator was a multiple of the other denominator (e.g., 2/3 + 1/9), both denominators were prime numbers (e.g., 2/7 + 3/5), or both denominators had a common divisor larger than one (e.g., 5/6 + 3/8). Self-reports, accuracy, and response times confirmed that participants adapted their strategy use according to problem type. We analysed the number of eye fixations on each fraction component, as well as the number of saccades (rapid eye movements) between fixations on components. We found that participants predominantly processed the fraction components separately rather than processing the overall fraction magnitudes. Alternating between the two denominators appeared to be the dominant process, although in problems with common denominators alternating between numerators was dominant. Participants rarely used diagonal saccades in any of the problems, which would indicate cross-multiplication. Our findings suggest that adults adapt their cognitive processes of fraction addition according to problem type. We discuss the implications of our findings for numerical cognition and mathematics education, as well as the limitations of our current understanding of eye movement patterns.

Keywords: strategy use, fraction addition, mental calculation, eye movements, saccades

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Proficiency with fractions is important for everyday life as well as for further mathematical learning (Bailey, Hoard, Nugent, & Geary, 2012; Booth & Newton, 2012; Siegler et al., 2012; Torbeyns, Schneider, Xin, & Siegler, 2015). Yet, learning to work with fractions poses great challenges for many students (e.g., Carraher, 1996; Mazzocco & Devlin, 2008; Siegler & Pyke, 2013; Stafylidou & Vosniadou, 2004; Vamvakoussi & Vosniadou, 2004). To better understand what makes fractions difficult, current research is attempting to unravel the cognitive processes involved in solving fraction problems (Siegler, Fazio, Bailey, & Zhou, 2013). To that end, some studies have employed eye tracking in addition to other methods of analysis because eye movements are believed to reflect cognitive processes (Grant & Spivey, 2003). However, to the best of our knowledge, all previous eye tracking studies focused on fraction processing have included only fraction comparison problems, in which participants were asked to choose the larger of two fractions (Huber, Moeller, & Nuerk, 2014; Hurst & Cordes, 2016; Ischebeck, Weilharter, & Körner, 2016; Obersteiner et al., 2014; Obersteiner & Tumpek, 2016). To date, little is known about the cognitive processes in more complex fraction arithmetic, such as fraction addition. Therefore, the current study investigates adults' cognitive processes during fraction addition, using measures of accuracy, response times, and eye movements.

Cognitive Processing of Fractions

Common fractions[i] are composed of two natural number components, the numerator and the denominator (e.g., 3/8 has a numerator of 3 and a denominator of 8). Thus, from a cognitive perspective, it is possible to process a fraction either *componentially*—as two separate whole numbers (3 and 8)—or *holistically*—as one (rational) number with one overall magnitude (i.e., the numerical value of 3/8). This distinction between componential and holistic processing is useful to understand why people struggle with solving fraction problems: many of the mistakes students make in fraction problems seem to be due to their reliance on componential processing in problems that require holistic processing, a phenomenon known as the "whole number bias" or "natural number bias" (Ni & Zhou, 2005; Vamvakoussi, Van Dooren, & Verschaffel, 2012; Van Hoof, Lijnen, Verschaffel, & Van Dooren, 2013). For example, when a representative sample of 13-year-old American school students were asked to choose, without calculation, their best estimate of the addition problem 12/13 + 7/8, the majority of them chose 19 (the sum of the numerators) or 21 (the sum of the denominators), and only 24% chose the correct answer of 2 (Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1981). Presumably, the majority of students engaged in componential rather than holistic processing while working with these two fractions. There is also evidence that in fraction addition problems, componential addition (i.e., adding the numerators and denominators separately, e.g., 1/2 + 2/3 = 3/5) without holistic reasoning is students' single most frequent mistake (Behr, Wachsmuth, & Post, 1985; Brown & Quinn, 2006; Eichelmann, Narciss, Schnaubert, & Melis, 2012; Padberg, 2009).

Given students' apparent difficulty processing fractions holistically, research has investigated whether our cognitive system is capable of processing fractions in a holistic manner, and if so, to what extent people make use of holistic processing of fractions. Many studies address these questions by exploring how educated adults and academic mathematicians solve number comparison problems (Bonato, Fabbri, Umiltà, & Zorzi, 2007; Meert, Grégoire, & Noël, 2009, 2010a, 2010b; Obersteiner, Van Dooren, Van Hoof, & Verschaffel, 2013; Schneider & Siegler, 2010). Based on measures of accuracy and response times, a major conclusion from this line of research is that people are—in principle—able to process fractions holistically, although not in an automatized manner (Meert et al., 2009, 2010a; Obersteiner, Van Dooren, et al., 2013; Schneider & Siegler, 2010). Moreover, studies suggest that while both componential and holistic processes are involved in fraction comparison problems of any type, the extent to which each process occurs depends on the specific strategy an individual uses (Meert et al., 2010a). Strategy choice in turn appears to depend on the affordances of the specific problem at hand (Alibali & Sidney, 2015; Fazio, DeWolf, & Siegler, 2016; Obersteiner, Van Hoof, & Verschaffel, 2013). When comparing two fractions that have common denominators (e.g., 3/7 vs. 5/7) or common numerators (e.g., 3/7 vs. 3/5), people rely predominantly on componential comparison strategies (i.e., they compare the unequal fraction components only), and these strategies include componential processing more so than holistic processing. In contrast, when comparing two fractions without common components (e.g., 3/7 vs. 5/8),

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people rely more strongly on the overall fraction magnitudes, inclining them towards more holistic processes than componential processes.

A limitation of the research described so far is that it relies on accuracy and response time measures, which are distal measures of cognitive processes since they do not capture potential differences in *how* people solve problems. It is therefore desirable to extend this research through more proximal measures. Self-reports may be a more proximal and straightforward way to assess strategy use (Clarke & Roche, 2009; Ericsson & Simon, 1993; Fazio et al., 2016), but the validity and reliability of self-reports are limited (Robinson, 2001) because people are not always aware of the strategies they used or may struggle to describe their strategies. Moreover, it is presumably difficult to describe the exact cognitive processes used if problem solving is completely or partly automatized, as might be the case in processing whole number components in fraction problems (Mock, Huber, Klein, & Moeller, 2016). Finally, asking participants to report their strategies may affect the way they solve a problem (Kirk & Ashcraft, 2001). Although it is impossible to directly assess cognitive processes, eye tracking is a less invasive and, compared to other methods, more direct measure of strategy use, since it arguably allows "on-line" assessment of the information participants are processing (Mock et al., 2016).

Eye Movements as a Measure of Cognitive Processes in Fraction Addition

During visually presented cognitive tasks, eye movements are assumed to correspond to mental operations (Grant & Spivey, 2003). The eye-mind assumption and the immediacy assumption are founded on the premise that the location and duration of eye fixations correspond to the content of the information being processed and the time needed to process it (Just & Carpenter, 1980, but see Anderson, Bothell, & Scott, 2004, for a critical view). An increasing number of studies use eye tracking successfully to detect strategy use during mathematical problem solving (e.g., Green, Lemaire, & Dufau, 2007; Roy, Inglis, & Alcock, 2017; Schneider et al., 2008; Sullivan, Juhasz, Slattery, & Barth, 2011).

Recent studies used eye tracking to investigate strategy use and the cognitive processes involved in fraction comparison (Huber et al., 2014; Hurst & Cordes, 2016; Ischebeck et al., 2016; Obersteiner et al., 2014; Obersteiner & Tumpek, 2016). Evaluating how long and how often people fixate on specific fraction components, these studies concluded—in line with earlier findings relying on measures of accuracy and response times (see previous section)—that adults use both componential and holistic processing in fraction comparison, and that the extent to which each process occurs depends on problem type. Componential processes are more common in comparison problems featuring common denominators, while holistic processes are more common in comparison problems lacking common components.

Recent eye tracking research also explored whether individual fraction components are more or less difficult to process when solving fraction comparison problems. Huber et al. (2014) suggested that processing fraction denominators requires more cognitive effort (as measured by the number of fixations) than processing fraction numerators. The reason denominators may be more difficult to process could be that while the size of the numerator corresponds to the overall size of the fraction (i.e., increasing the numerator increases the fraction value), the denominator is inversely related to the overall size of the fraction (i.e., increasing the denominator decreases the overall size of the fraction), making it more demanding to process. However, evidence from other studies points in the opposite direction (Hurst & Cordes, 2016) or suggests that the role of the denominator depends on the affordances of the specific problem type (Obersteiner & Tumpek, 2016). The difference in processing demands between numerators and denominators may be more pronounced in fraction addition, where the fraction denominators play a particularly important role. The reason is that in fraction addition, the denominator provides important information about which strategy is most efficient, and standard strategies include calculating the least common denominator, which is presumably a demanding process.

Strategies and Cognitive Processes in Fraction Addition

The results from studies of fraction comparison reviewed above (that holistic and componential fraction processing depends on whether fractions have common components) may not be directly transferable to fraction addition. The cognitive processes involved in fraction addition might differ substantially from those in fraction comparison because addition problems require the use of different strategies than fraction comparison problems. For example, it is not evident that people use holistic processing in fraction addition. While holistic processing can be beneficial to estimate or to double-check the result of an addition problem,[ii] it is not absolutely necessary. In fact, the standard addition algorithm taught in schools relies only on manipulating the fraction components and does not involve holistic processing. This standard strategy is generally valid for adding any two fractions (e.g., 5/6 + 3/4). It includes three steps:

- Step 1: Find the least common denominator[iii] of the two fractions (e.g., the least common denominator of 6 and 4 is 12).
- Step 2: Multiply the numerators and the denominators of both fractions so the fractions will have like denominators (e.g., 5/6 = (5 x 2)/(6 x 2) = 10/12 and 3/4 = (3 x 3)/(4 x 3) = 9/12).
- Step 3: Add the new numerators and use the least common denominator as the denominator (e.g., 10/12 + 9/12 = 19/12).

In the following, we refer to fraction addition problems that require all of these steps, and that are not among the special cases described below, as *Standard*.

Not every fraction addition problem requires these three steps. There are at least three special cases of fraction addition problems. The following three types of problems have specific affordances, in the sense that one or more of the steps described above are especially easy to carry out or can be skipped entirely.

If one of the two fraction denominators is a multiple of the other denominator (e.g., 5/6 + 7/12; hereafter referred to as a *MultiDenom* problem), Step 1 is especially easy to carry out, because the larger denominator is also the least common denominator. This may reduce the cognitive effort required to find the least common denominator. Moreover, Step 2 is especially simple because one needs to multiply only the components of the fraction with the smaller denominator, not of the fraction with the larger denominator. For example: 5/6 + 7/12 = (5 x 2)/(6 x 2) + 7/12 = 10/12 + 7/12 = 17/12.

If the two denominators are prime numbers[iv] (that are different from one another, e.g., 2/3 + 5/7; hereafter referred to as a *PrimeDenom* problem), the least common denominator is the product of the two denominators. Although multiplying the denominators can result in a relatively large number (compared to the least common denominator in the Standard problem type), multiplying the denominators reduces the effort required to determine common factors of the two denominators (because there are no common factors) and the calculation effort required to determine the least common denominator. Moreover, Step 2 is especially easy to carry out because one needs to multiply each numerator with the denominator of the other fraction (i.e., to cross-multiply). It is therefore not necessary to keep track of the numbers with which one has to multiply the two fractions' components. For example: 2/3 + 5/7 = (2 x 7 + 3 x 5)/(3 x 7) = (14 + 15)/21 = 29/21.

If the two fractions have like denominators (e.g., 7/8 + 3/8; hereafter referred to as a *LikeDenom* problem), one can skip Steps 1 and 2 altogether. It is sufficient to simply add the numerators and maintain the common denominator. For example: 7/8 + 3/8 = 10/8.

The Present Study

This study assesses eye movement patterns in different types of fraction addition problems. The aims of this study are to investigate how strongly adults rely on componential and holistic processing when solving fraction addition problems; whether adults adapt their strategies according to problem type; how the cognitive processes differ between problem type; and how adults distribute their attention over the two fraction components.

Our study extends previous research in at least five ways. First, while previous research largely focuses on fraction comparison, we study more complex fraction addition problems, thus providing a broader insight into the cognitive mechanisms of fraction processing in the context of fraction arithmetic. To our knowledge, no studies to date have assessed eye movements in fraction addition. Second, we not only analyse the number of eye fixations on the fraction components but also the saccades (rapid eye movements) between the fixations on these components. Arguably, saccades are a better measure of cognitive processes than fixations alone because they track which fraction components participants fixate on in succession. As in previous research (Ischebeck et al., 2016; Obersteiner & Tumpek, 2016), we assume that saccades between the two fraction numerators and saccades between the fraction denominators indicate component processing, while saccades between the numerator and the denominator of one fraction indicate holistic processing. Third, in an approach similar to the one employed by Obersteiner and Tumpek (2016), we analyse the relative frequencies of different saccades within each problem type, rather than the distribution of specific saccades over different problem types as reported by Ischebeck et al. (2016). The former analysis provides a general pattern of saccades in different problem types and is thus a better basis for interpreting differences in cognitive processes between problem types. Fourth, we create addition problems of different types while controlling for potentially confounding factors such as the sizes of the components (see Methods section). Fifth, we present the addition problems on a computer screen in an almost natural manner to increase external validity. More specifically, the spatial distances between number symbols are just large enough to reduce the possibility of people identifying fraction components without fixating on them (using peripheral vision) and yet small enough to make the fractions look natural, as they might appear in a textbook. This was not always the case in previous studies. For example, in an attempt to eliminate peripheral vision, Ischebeck et al. (2016) used very large distances between the fraction components. In that study, the distances between the numerators and the denominators of each fraction were the same as the distances between the two numerators and between the two denominators, so that the stimuli looked more like four separate numbers rather than a pair of fractions.

We used the four types of addition problems (Standard, MultiDenom, PrimeDenom, LikeDenom) described above, for which the most efficient strategies include different processes (see section entitled Strategies and Cognitive Processes in Fraction Addition). In addition to eye movements, we also analysed accuracies and response times. We expected that both accuracy and response times would differ significantly between problem types, with mean accuracy decreasing (Hypothesis 1a) and response times increasing (Hypothesis 1b) in the following order: LikeDenom, MultiDenom, PrimeDenom, Standard. We expected that accuracies and response times would reflect the increasing difficulty level of these problem types, a result of the increasing cognitive effort required to reach a solution.

In regard to eye movement patterns, the first parameter we analysed was the number of fixations on numerators and denominators. We hypothesized that problem type would interact with the relative number of fixations on numerators and denominators (Hypothesis 2). In LikeDenom problems, we expected the number of fixations on numerators to be higher than those on denominators because in these problems it is sufficient to add the numerators without performing any operations on the denominators. In contrast, we expected that there would be more fixations on denominators than on numerators in problems of all other types because the denominators determine the most efficient strategy for solving the problem (see above). Moreover, participants would pay more attention to denominators because these problems require finding a common denominator, which is presumably a demanding process. In MultiDenom and PrimeDenom problems, we expected that the difference between fixations on numerators and denominators would be less pronounced than in Standard problems because of the reduced effort to find a common denominator.

The saccades between the fraction components were of primary interest. We used an approach similar to the one employed by Obersteiner and Tumpek (2016) and counted the number of the six types of saccades that connect the four fraction components: 1) saccades between the numerators, 2) saccades between the denominators, 3) saccades between the left numerator and the left denominator, 4) saccades between the right numerator and the right denominator, 5) saccades between the left numerator and the right denominator, and 6) saccades between the left denominators and the right numerator. We considered saccades between corresponding fraction components (Types 1 and 2) as indicators of *componential* processing, and saccades between the numerator and denominator of the same fraction (Types 3 and 4) as indicators of *holistic* processing. Diagonal saccades (Types 5 and 6) may indicate *cross-multiplication* processes. Although cross-multiplying is also a sort of componential processing (as opposed to holistic processing), we only use the term "componential" to refer to processes that connect the *corresponding* fraction components (i.e., both numerators, or both denominators) rather than different components (i.e., one numerator and one denominator). This terminology is in line with the literature on fraction processing that has often not considered diagonal processing (Huber et al., 2014; Ischebeck et al., 2016; Meert et al., 2009; but see Obersteiner & Tumpek, 2016).

As a general indicator of difficulty level, we expected to find differences between problem types in the total number of the saccades described above, similar to accuracy and response times. The total numbers of saccades should increase moving from LikeDenom to MultiDenom to PrimeDenom to Standard problems (Hypothesis 3). Since previous research demonstrated that people use a variety of saccades in solving fraction comparison problems (Ischebeck et al., 2016; Obersteiner & Tumpek, 2016), we did not establish hypotheses concerning the relative numbers of all six types of saccades for each problem type. However, we anticipated that within each problem type, the majority of saccades would be componential because holistic reasoning is not required to solve addition problems. More specifically, we expected that in LikeDenom problems, the number of saccades between numerators would be higher than the numbers of saccades in each of the other categories (Hypothesis 4a) because adding the numerators is sufficient in these problems. For MultiDenom and Standard problems, we anticipated that the saccades between denominators would be highest (Hypotheses 4b and 4c, respectively) because in these types of problems, determining the least common denominators should draw increased attention to denominators. In the case of PrimeDenom problems, we expected that the diagonal saccades (i.e., Types 5 and 6) would be more frequent than in other types of problems (Hypothesis 4d). These saccades indicate cross-multiplication processes, which are efficient to solve PrimeDenom problems. Note that we did not expect diagonal saccades to be more frequent than saccades of other types *within* PrimeDenom problems, because not all participants would engage in cross-multiplication. Moreover, Obersteiner and Tumpek (2016) found that adults rarely use cross-multiplication during fraction comparison although this strategy is viable.

Methods

Participants

The participants in this study included 28 adults (9 females and 19 males). Their mean age was 24.3 years (*SD* = 2.3). They were recruited from three universities in a large city in Germany. The participants were enrolled in a variety of study programs, most of them in the STEM (science, technology, engineering, mathematics) fields. It is therefore reasonable to assume that our participants had no particular expertise in fraction arithmetic but were well able to solve the fraction addition problems in our experiment. All participants reported to have normal or corrected-to-normal vision.

Fraction Addition Problems

We constructed 40 fraction addition problems of four different types. In LikeDenom problems (10 problems), the two fractions had the same denominator. In MultiDenom problems (10), the denominator of one fraction was a multiple of the denominator of the other fraction. In PrimeDenom problems (10), the denominators were unequal prime numbers. Standard problems (10) did not belong to any of the before-mentioned types. In these problems, the denominators had a common divisor larger than one.

To reduce processes unrelated to addition, all fractions were presented in their simplified form. We only included proper fractions (i.e., all fraction values were smaller than 1). The numerators of all fractions were one-digit numbers, and the denominators were all one- or two-digit numbers smaller than 20, not including ten. The results of the addition problems were never equal to one. The problems were designed such that several problem features were comparable between problem types. Among these features were the average magnitudes of the numerators (problem type averages ranging from 4.25 to 4.60), of the denominators (8.20–11.05), of the two fractions (0.45–0.67), and of the results of the additions (0.90–1.34). In half of the problems, the first addend of the addition problem was larger, and in the other half, the second addend was larger. For a complete list of the addition problems used in this study, see the Appendix.

Procedure

The participants worked on the problems individually in a quiet room at the university. They sat in front of a 22-inch computer screen connected to a remote SMI eye tracker with a sampling rate of 500 Hz. Before the experiment began, participants performed a five-point calibration and solved a practice problem. The participants were instructed that they would see fraction addition problems on the screen and that they should try to verbally provide the correct answer as quickly and as accurately as possible. If they felt unable to solve a problem, they should say “continue.” The experimenter recorded the participants’ responses. As in the study conducted by

Green et al. (2007), we used verbal responses rather than key press responses because we wanted to avoid the participants looking at the keyboard (resulting in potential loss of eye tracking data) to type in their responses. The problems appeared in the centre of the screen, with fraction bars of approximately 5 cm in width and fraction numerators and denominators each of approximately 4 cm in height. The distance between the two fraction bars was approximately 8 cm. The plus symbol was centred between the two fractions. After the participants provided their responses, the experimenter pressed a key and a fixation cross appeared in the centre of the screen for 1 second. Afterwards, the next problem appeared. The response time for a problem was defined as the time between the appearance of the problem on the screen and the moment the experimenter pressed the key. The problems appeared in random order.

After the eye tracking session, we asked participants to record on a worksheet the strategies they had used to solve fraction addition problems of each type. The worksheet illustrated one sample problem of each of the four types that participants had worked on during the eye tracking experiment. We wanted to determine whether participants actually made use of the strategies we expected to be most efficient for the four different problem types.

Data Analysis

Although eye movements are the main focus of this study, we first analysed accuracy and response times. We used analyses of variance (ANOVA), a priori contrasts, and post hoc *t*-tests for analysing systematic differences between problem types. To determine effect sizes, we calculated partial eta squared (η_p^2) for ANOVAs and Cohen's *d* for *t*-tests (e.g., Field, 2013). To analyse eye movements, we defined four equally sized rectangular areas of interest that covered the numerator or the denominator of the left or the right fraction. Each area of interest began at the fraction bar of the respective fraction and measured 353 pixels wide by 430 pixels high. All analyses were conducted with IBM SPSS 24 (IBM Corporation, 2016), and effect sizes were calculated with G*Power 3 (Faul, Erdfelder, Lang, & Buchner, 2007).

Results

Self-Reported Strategies

Table 1 provides an overview of the strategies participants reported using for each of the four problem types. It also notes the absolute and relative frequencies of reported use. For LikeDenom problems, almost all participants reported just adding the numerators, the result we expected. For MultiDenom problems, approximately two thirds of participants reported relying on a strategy similar to the standard algorithm, although they multiplied *one* instead of two fractions, which was also the strategy we expected. However, a relatively large number of participants (approximately one fifth) reported using the full standard algorithm, including multiplication of *both* fractions, for these problems[v]. For both PrimeDenom and Standard problems, the large majority of participants reported using the standard algorithm. While we expected this strategy in response to Standard problems, it is surprising that only two participants reported using cross-multiplication in PrimeDenom problems. This may suggest that the majority of participants were not aware that in PrimeDenom problems, the denominators are the numbers with which one has to multiply the other fraction. On the other hand, cross-multiplication is just a special way of finding common denominators and participants may not explicitly report using it even if they did.

Table 1

Self-Reported Strategies per Problem Type

Problem Type / Strategy	Frequency	Percent
LikeDenom		
Check denominators, then add numerators	26	93
Just add numerators	1	4
Just add fractions	1	4
MultiDenom		
Find common denominator, multiply one fraction, add numerators	17	61
Find common denominator, multiply both fractions, add numerators	6	21
Multiply one fraction, then add numerators	5	18
PrimeDenom		
Find common denominator, multiply both fractions, add numerators	24	86
Find common denominator	2	7
Find common denominator, add cross-product	1	4
Add cross-product	1	4
Standard		
Find common denominator, multiply both fractions, add numerators	22	79
Find common denominator, multiply both fractions	3	11
Find common denominator, add numerators	1	4
Find common denominator	1	4
Multiply both fractions, add numerators, find common denominator	1	4

Note. When referring to common denominators, some of our participants actually used the term “least common denominator”, while others said “common denominator” (without specifying whether or not they meant the “least” common denominator). We do not distinguish these responses in the table because it is not relevant for our analysis.

More generally, it is important to note that participants’ responses may not be completely reliable because some participants did not report all the steps necessary to solve these addition problems. For example, in response to MultiDenom problems, five participants did not report finding a common denominator. Rather, the first step these participants reported was multiplying one fraction, a step for which finding a common denominator is, however, a prerequisite. We assume that participants did not explicitly mention finding a common denominator because they assumed that this step was included in the first step they described (multiplying one fraction). Likewise, the strategy “find common denominator” in response to Standard problems (as reported by one participant) is clearly incomplete.

It is also noteworthy that none of our participants mentioned using holistic reasoning about the fraction magnitudes, which may have been a way to check whether or not their result is reasonable. However, we assume that even if participants used holistic reasoning, they might not have considered such reasoning part of their strategy and thus did not report it.

As a whole, however, these reports confirm our assumption that the problems we presented encouraged different processes and that the large majority of our participants were aware of the different affordances of these problem types.

Accuracy and Response Times

Table 2 shows mean accuracy and mean response time for correctly solved problems of all four types. As expected (Hypothesis 1a), there were substantial and highly significant differences in mean accuracies between problem types, $F(3, 81) = 29.91$, $p < .001$, $\eta_p^2 = .69$. These differences were ordered as expected, with accuracy being highest for LikeDenom problems, followed by problems of types MultiDenom, PrimeDenom, and Standard. All pairwise comparisons between problem types were significant (all $p < .05$), excepting the difference between problems of types PrimeDenom and Standard ($p = .528$). Analyzing response times, we excluded incorrectly solved problems (19.8%) and problems with response times that deviated more than two standard deviations from the sample mean per problem type (another 4.5%). There were significant differences in response times between problem types, $F(3, 81) = 137.93$, $p < .001$, $\eta_p^2 = .84$, with response times increasing in the same order as the accuracies decreased (from LikeDenom to MultiDenom to PrimeDenom to Standard). All pairwise comparisons between problem types were highly significant (all $p < .001$), supporting Hypothesis 1b.

Table 2

Mean Accuracy and Response Times for Correctly Solved Problems per Type

	Accuracy (%)		**Response Time (ms)**	
Problem Type	***M***	***SD***	***M***	***SD***
LikeDenom	96.1	5.7	4262	724
MultiDenom	91.1	11.0	8683	1958
PrimeDenom	70.4	23.2	23712	7916
Standard	63.2	25.5	31564	10143
All	80.2	22.7	17055	12829

Eye Tracking Data

In the analyses of eye movement data reported in the remainder of this section, we excluded four participants due to low calibration quality. This exclusion reduced the sample size to 24.

Number of Fixations

As an indicator of the relative importance of the numerators and denominators, we analysed the average number of fixations on numerators and denominators per problem for each problem type. Table 3 displays these results. There was a significant effect of *problem type* on the number of fixations, $F(3, 69) = 102.04$, $p < .001$, $\eta_p^2 = .82$, with the lowest number of fixations appearing in LikeDenom problems, followed by MultiDenom, PrimeDenom, and Standard problems. There was also a significant effect of *component*, $F(1, 23) = 99.37$, $p < .001$, $\eta_p^2 = .81$, indicating that there were generally more fixations on denominators than numerators. Notably, there was also a significant interaction between *problem type* and *component*, $F(3, 69) = 85.35$, $p < .001$, $\eta_p^2 = .79$, indicating that the difference in the number of fixations between numerators and denominators depended on problem type. Post-hoc *t*-tests with paired samples illustrated that there were more fixations on numerators than denominators in LikeDenom problems, $t(23) = 4.37$, $p < .001$, $d = 0.90$. In contrast, there were more fixations on denominators than numerators in all other problem types, with the effect sizes increasing from MultiDenom problems, $t(23) = 2.98$, $p = .007$, $d = 0.61$, to PrimeDenom problems, $t(23) = 5.61$, $p < .001$, $d = 1.14$, to Standard problems, $t(23) = 10.56$, $p < .001$, $d = 2.16$. The important role of the denominators for problems with unlike denominators was as expected (Hypothesis 2), but the data reveals the additional finding that the dominant role of the denominators increased dramatically as problem difficulty increased.

Table 3

Numbers of Fixations on Numerators and Denominators per Problem, for Each Problem Type

	Number of Fixations			
	Numerators		Denominators	
Problem Type	***M***	***SD***	***M***	***SD***
LikeDenom	2.78	0.77	2.08	0.51
MultiDenom	4.84	1.70	5.56	1.32
PrimeDenom	10.30	4.94	13.74	4.12
Standard	10.69	5.53	22.86	8.72
All	7.15	5.10	11.06	9.38

Number of Saccades

We also analysed the number of saccades between each of the four AOIs. First, it is notable that the total number of saccades varied substantially between the four problem types, ranging from only seven saccades in LikeDenom problems to 30 saccades in Standard problems. As expected, the number of saccades for the other two problems types fell in between, with an increase from LikeDenom to MultiDenom, PrimeDenom, and Standard problems (see Table 4). These differences in the number of saccades between problem types were significant, $F(3, 69) = 49.29$, $p < .001$, $\eta_p^2 = .68$, as were the pairwise comparisons (all $p < .05$), supporting Hypothesis 3.

Table 4

Numbers of Saccades per Problem, for Each Problem Type

	Number of Saccades	
Problem Type	***M***	***SD***
LikeDenom	6.78	1.80
MultiDenom	13.04	4.99
PrimeDenom	23.84	12.03
Standard	29.31	15.49
All	18.24	13.38

Proportions of Different Types of Saccades

To analyse the proportions of the different types of saccades within each problem type, we first calculated per problem the percentage of each saccade type relative to the total number of relevant saccades on that problem. We averaged this number across the ten problems of each type for each participant and then across participants. Figure 1 shows the proportions of all six types of saccades per problem type.[vi]

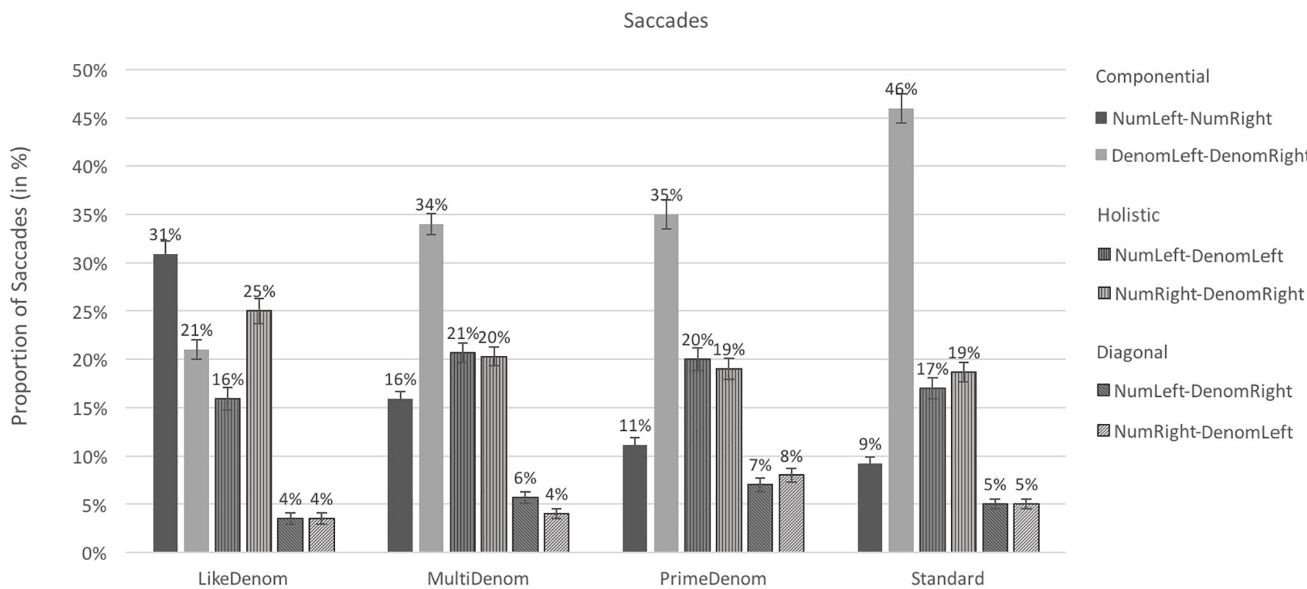

Figure 1. Proportions of saccades within each problem type.

Note. Error bars indicate standard errors of the mean. Percentages refer to the distribution within each problem type. The sum of percentages within each problem type may deviate slightly from 100 due to rounding.

The figure demonstrates that all types of saccades occurred in problems of any type. As expected (Hypothesis 4a), the saccades between numerators played the dominant role in LikeDenom problems, with their proportion being highest among the different saccades of interest. Particularly, they were more frequent than saccades between denominators, $t(23) = 3.32$, $p = .003$, $d = 0.71$. In contrast, saccades between denominators were by far the most frequent types of saccade in all other problem types, and their frequency increased in order of problem difficulty (from MultiDenom to PrimeDenom to Standard). Regarding MultiDenom and Standard problems, repeated measures ANOVAs and post-hoc comparisons confirmed our predictions (Hypotheses 4b and 4c) that the saccades between denominators were more frequent than saccades of any other type (MultiDenom: $F(5, 115) = 69.64$, $p < .001$, $\eta_p^2 = .75$; Standard: $F(5, 115) = 143.37$, $p < .001$, $\eta_p^2 = .86$; with $p < .001$ for all comparisons between saccades connecting denominators and any other saccades, in both problem types). For all problem types, diagonal saccades occurred only rarely, suggesting that cross-multiplication was not frequently applied in solving these addition problems. Even in PrimeDenom problems, for which cross-multiplication would be most efficient, saccades along the two diagonals accounted for only 7% and 8%, respectively, of the saccades of interest. Problem types did not differ significantly in terms of the frequencies of diagonal saccades between the left numerator and the right denominator, $F(3, 92) = 1.37$, $p = .257$. However, they did differ significantly in the frequencies of diagonal saccades between the right numerator and left denominator, $F(3, 92) = 4.95$, $p = .003$, $\eta_p^2 = .14$. Post hoc comparisons showed that saccades of this type were more frequent in PrimeDenom problems than in LikeDenom problems, $t(23) = 3.16$, $p = .004$, $d = 0.67$, and in PrimeDenom problems than MultiDenom problems, $t(23) = 2.98$, $p = .007$, $d = 0.60$, with no significant difference between PrimeDenom problems and Standard problems. These analyses only partially support Hypothesis 4d, which would have assumed that both types of diagonal saccades occur generally more frequently in PrimeDenom problems compared to any other problem type.

It is also notable that for all problem types, holistic saccades (i.e., those between the numerator and the denominator of each fraction) were relatively common, regardless of problem type. These saccade types occurred nearly as often in all problem types, except for LikeDenom problems, where saccades between the numerator and the denominator of the right fraction occurred more often than those between the numerator and denominator of the left fraction.

Discussion

The aim of this study was to explore the cognitive processes involved in fraction addition by adults, thereby extending previous research on fraction comparison (Huber et al., 2014; Hurst & Cordes, 2016; Ischebeck et al., 2016; Obersteiner et al., 2014; Obersteiner & Tumpek, 2016). We were particularly interested in the extent to which adults use componential processing, holistic processing, and cross-multiplication, and how these processes relate to strategy use in different problem types.

Strategy Use in Different Problem Types

The addition problems in our study differed with respect to the strategies that were thought to be most efficient for solving them. Participants' self-reports largely confirmed that they adapted their strategies to the affordances of the different problem types. Response times and accuracy also reflected the different affordances of problems of different types. Participants were particularly correct and fast in adding fractions with like denominators, where the only required process was to add the numerators. For other addition problems, difficulty is presumably determined by the cognitive effort required in the different solution steps, namely finding a common denominator, multiplying the fraction components, and adding the numerators. In line with our analysis of these steps (see section entitled Strategies and Cognitive Processes in Fraction Addition), accuracy decreased and response times increased from MultiDenom problems to PrimeDenom problems to Standard problems. Overall eye movement parameters such as the numbers of fixations and the total numbers of saccades per problem also increased from one problem type to another in the same order, supporting our basic assumption that eye movement patterns correspond closely to the problem solvers' cognitive processes. We can conclude that participants adapted their strategies to the affordances of the problems, which is in line with findings from fraction comparison studies (Fazio et al., 2016; Huber et al., 2014; Ischebeck et al., 2016; Obersteiner & Tumpek, 2016).

Cognitive Processes Depending on Problem Type

As different strategies should require different cognitive processes, the results discussed in the previous section already suggest that different processes are involved in solving fraction addition problems of different types. Our analysis of the proportions of saccade types within each problem type allowed for a more detailed analysis of these processes. We had expected that, within each problem type, the majority of saccades would be componential, because fraction addition does not necessarily require holistic processing of the fraction magnitudes. The results show that for fractions with like denominators, the (componential) saccades between numerators account for the majority of saccades within this problem type. For all other problem types, the most frequently occurring saccades are (componential) saccades between the denominators. Although holistic saccades occur in all problems types, these saccades (counted separately for the two fractions within a problem) never represented the majority of saccades. These results offer further support for the assumption that adults increase their reliance on holistic processing of fractions only when this is required for solving the problem, presumably because holistic processing is more demanding than componential processing. For example, in the study by Obersteiner and Tumpek (2016), holistic saccades account for the majority of saccades within comparison problems that require holistic processing (those without common components) but not in other problems that do not require holistic processing (those with common components). Our conclusion is also in line with accuracy and response time patterns documented in earlier fraction comparison studies (Meert et al., 2010a, 2010b; Obersteiner, Van Dooren, et al., 2013; Schneider & Siegler, 2010).

Although holistic saccades for the left or the right fraction did not represent the majority of saccades for any problem type, the proportions of holistic saccades were remarkably high, given that the addition problems did not require holistic processing. In fact, the holistic saccades for the left and the right fraction *together* accounted for approximately 40 percent of all saccades in each problem type. This proportion is only a little less than the proportion of holistic saccades in fraction comparison problems reported by Obersteiner and Tumpek (2016) (42–52%, depending on problem type). One explanation could be that participants *did* use holistic reasoning to some extent, for example to double-check their results. Another explanation is that these saccades do not solely reflect holistic reasoning but also more general processes such as reading. Ischebeck et al. (2016) assumed that the first four fixations in their fraction comparison problems represented reading processes. While this seems to be a reasonable assumption, one could also argue that people already reason about number magnitudes while reading, making it hard to disentangle reading from task-relevant numerical processes. Further research is needed to separate the different processes in fraction problem solving, such as reading and calculating. One approach could be to present identical stimuli in experimental conditions (calculation) and in control conditions (reading only).

Somewhat unexpectedly, diagonal saccades did not play a major role in fraction addition problems, not even in those with prime denominators. Although cross-multiplication would be an efficient strategy in these problems, the participants in our study did not seem to make extensive use of this strategy. This finding corresponds to our participants' self-reports, in which only two participants reported using cross-multiplication. It is also in line with the study by Obersteiner and Tumpek (2016), in which diagonal saccades rarely occurred in fraction comparison.

Processing of Numerators and Denominators

We have assumed that fraction denominators play a key role in fraction addition problems because strategy choice depends largely on the relation between the denominators and because finding a common denominator —which was required in three fourths of the problems in our study—is a relatively demanding step (see section entitled Strategies and Cognitive Processes in Fraction Addition). Thus, we expected that processing the denominators would be more demanding than processing the numerators in all problem types except for LikeDenom.

The number of fixations and the saccades analyses are in line with our predictions. In LikeDenom problems, people fixate less often on the denominators than on the numerators, and they switch less often between denominators than between numerators presumably because it is sufficient to add the numerators to find the result. In contrast, in all other problem types, the denominators require more attention than the numerators, as indicated by both fixations and saccades. While previous studies on fraction comparison were inconclusive about whether denominators are generally more difficult to process than numerators (Huber et al., 2014; Hurst & Cordes, 2016; Obersteiner & Tumpek, 2016), our data suggest that processing the denominator plays a key role in fraction addition. Thus, the relative difficulty of processing a fraction's numerator and denominator might depend on the specific affordances of the fraction problem.

Implications for Mathematics Education

It is well documented that school students who struggle with fraction problems rely heavily on componential processing and hardly employ holistic processing when working with fractions (see section entitled Cognitive Processing of Fractions). The specific cognitive processes of individual students are, however, less well understood. In further research, the eye movement patterns of educated adults identified in our study may help interpret eye movement patterns of students who process fractions ineffectively. For example, if a student solves a Standard fraction addition problem by purely relying on componential addition (i.e., adding the numerators and denominators separately), we would expect—relative to the pattern of our adult sample—larger proportions of saccades between numerators, and lower proportions of saccades between denominators. Moreover, the absence of holistic processing should be expressed by lower proportions of holistic saccades.

Assessing *holistic* processing of fractions may prove particularly relevant for mathematical learning because according to current standards in mathematics education, estimation and arguing about the reasonableness of a result are increasingly important skills (Common Core State Standards Initiative, 2010). Eye tracking could be used—in addition to measures of accuracy, response times, or verbal reports—to assess these processes more reliably. Of course, further research and technological development is necessary before eye movements may actually be used as an assessment tool in classrooms.

Limitations

A major limitation of our study is that eye movements may not be a perfectly valid measure of cognitive processes and strategy use in fraction addition, although previous research suggests that eye movements are a valid measure of strategy use in other mathematical problems such as number line estimation (Schneider et al., 2008). We used participants' self-reports to validate the assumption that our participants adapted their strategies to problem types. However, as in previous research (e.g., Ganor-Stern & Weiss, 2016), we assessed these self-reports in a separate session after the eye tracking experiment. To better match participants' self-reports to their eye movement patterns, it would be preferable to collect these reports on a trial by trial basis during (think-aloud) or right after (immediate recall) the actual problem solving process. This alteration may be challenging, though, because eye tracking tolerates only minimal head movement while providing verbal or written responses might induce head movement. Another alternative would be asking participants to verbalize their strategies while watching their own eye movements recorded in a prior eye tracking session (cued retrospective think-aloud). Although this method allows directly matching eye movements and self-reports, it poses the additional challenge for participants to interpret eye movement patterns.

Another limitation of our study is that our interpretations of eye movement patterns are tentative to some extent. The reason is that although these interpretations are based on theoretical considerations and empirical evidence from previous fraction comparison studies, our knowledge about how to interpret eye movements is still limited. For example, it is not completely clear if all saccades between the numerator and the denominator of a fraction actually indicate holistic fraction processing. As discussed earlier (see section entitled Cognitive Processes Depending on Problem Type), some of these numerator-denominator saccades may indicate reading processes. Moreover, these saccades may also indicate additive rather than multiplicative comparisons between numerators and denominators. This means that participants may reason about the difference rather than the quotient of a fractions' numerator and denominator, and thus may not process (holistic) fraction magnitudes. Further research is certainly necessary to better understand the relationship between specific cognitive processes and eye movement patterns.

Conclusion

Research suggests that adults compare fractions in both componential and holistic ways and that their strategies depend on the type of problem they encounter. Our study supports and extends these findings. Educated adults solve fraction addition problems mainly by focusing on and switching between the most informative fraction components. Whether these components are the numerators or the denominators depends on the specific type of addition problem at hand. In regard to methods, combining different measures such as self-reports, accuracy rates, response times, and eye movements is a promising way to gain insight into the cognitive processes of solving fraction problems.

Notes

i) Note that we only consider positive fractions here.

ii) For example, one might reject that 2/3 + 3/4 = 5/7, a result emerging from componential addition, without actually calculating the sum of the fractions, by arguing that adding two fractions larger than a half should result in a number larger than one, which 5/7 is not.

iii) It is actually not necessary to use the *least* common denominator to add fractions. Rather, *any* common denominator, whether the smallest or not, can be used. However, the standard algorithm suggests using the least common denominator to keep the numbers as small as possible.

iv) The strategy described for problems with prime denominators does not necessarily require prime denominators. This strategy is actually valid for any fraction addition problem. However, in problems with denominators that are not prime numbers, multiplying the denominators can result in large numbers, so that it is often more reasonable to find the least common denominator (as described in Standard problems), rather than multiplying the denominators.

v) Note that these participants' responses imply that they used a common denominator that was larger than both denominators of the two given fractions, rather than the larger of the two denominators. For example, participants may have multiplied both denominators, although they did not explicitly mention doing so.

vi) We ran a cluster analysis to identify potential subgroups of participants that might differ in their saccade patterns (and thus in their cognitive processes). This analysis showed, however, that we can consider our sample as a homogeneous group in terms of saccade patterns.

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Competing Interests

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Appendix: List of Fraction Pairs of Each Type

Like Denominators (LikeDenom)	Multiple Denominators (MultiDenom)	Prime Denominators (PrimeDenom)	Standard
3/5 + 4/5	3/20 + 4/5	3/11 + 4/5	5/6 + 7/8
2/7 + 1/7	2/7 + 1/14	2/7 + 1/11	7/12 + 4/9
5/8 + 7/8	7/18 + 4/9	3/5 + 6/7	1/16 + 7/12
7/9 + 4/9	1/6 + 7/12	2/3 + 8/13	4/9 + 8/15
3/7 + 6/7	9/16 + 3/8	4/5 + 8/11	5/18 + 1/12
1/12 + 7/12	4/5 + 8/15	1/2 + 8/17	3/14 + 1/6
2/13 + 8/13	5/18 + 1/3	6/19 + 1/2	5/8 + 3/14
9/16 + 3/16	7/8 + 1/2	2/3 + 9/13	7/18 + 3/4
4/15 + 8/15	2/3 + 8/15	9/17 + 1/3	5/6 + 3/16
5/18 + 1/18	3/4 + 5/12	5/7 + 2/3	1/4 + 9/14

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