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Adaptive Number Knowledge in Secondary School Students: Profiles and Antecedents

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Abstract

The present study aims to examine inter-individual differences in adaptive number knowledge in secondary school students. Adaptive number knowledge is defined as a well-connected network of knowledge of numerical characteristics and arithmetic relations. Substantial and relevant qualitative differences in the strategies and expression of adaptive number knowledge have been found in primary school students still in the process of learning arithmetic. We present a study involving 879 seventh-grade students that examines the structure of individual differences in adaptive number knowledge with students who have completed one year of algebra instruction. Results of a latent profile analysis reveal a model that is similar than was previously found in primary school students. As well, arithmetic fluency and the development of arithmetic fluency are strong predictors of adaptive number knowledge latent profile membership. These results suggest that adaptive number knowledge may be characteristic of high-level performance extending into secondary school, even after formal instruction with arithmetic concludes.

Keywords: adaptive number knowledge, arithmetic development, individual differences, latent profile analysis, adaptive expertise

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Adaptivity with arithmetic refers to the ability to solve a particular problem in the most efficient and appropriate way in a particular situation (Verschaffel, Luwel, Torbeyns, & Van Dooren, 2009). Adaptivity with arithmetic problem solving is argued to be dependent on three constraints (1) problem characteristics, such as the arithmetic relations that exist between the numbers, (2) situational characteristics, such as the amount of time pressure, the presence of mental or written representations, and the level of exactness needed, and (3) individual characteristics, such as competences to use different problem solving strategies and the flexibility to move between them. The present study aims to examine one aspect of these individual characteristics that underlie inter-individual differences in adaptivity with arithmetic, adaptive number knowledge.

Another of these individual characteristics, procedural flexibility had been described as the ability to switch between different problem solving strategies (e.g. Torbeyns, de Smedt, Ghesquière, & Verschaffel, 2009). However, procedural flexibility has been identified as only a portion of what individual characteristics of adaptivity may entail (e.g. Threlfall, 2009; Verschaffel et al., 2009). Instead, it may be just as important to examine at more general numerical and arithmetic knowledge that makes adaptivity with arithmetic problem solving possible (e.g. Canobi, Reeve, & Pattison, 2003). To ameliorate this gap, McMullen and colleagues (2016, 2017) have introduced the construct of adaptive number knowledge. Adaptive number knowledge is defined as a well-connected network of knowledge of numerical characteristics and arithmetic relations between numbers. Previously, individual differences in adaptive number knowledge has been described in late primary school students who are still in the process of developing their basic arithmetic knowledge and skills (McMullen et al., 2017). The present study aims to extend these findings by examining the nature of adaptive number knowledge in lower secondary school students, and for the first time, examine the mathematical and cognitive predictors of adaptive number knowledge.

Adaptive Number Knowledge

Adaptive problem solving strategies with whole-number arithmetic may be reliant on students' understanding of the numerical characteristics of the numbers presented in the problem and a rich understanding of the arithmetic relations between these numbers (Rathgeb-Schnierer & Green, 2013; Threlfall, 2009; Verschaffel et al., 2009). This is expected to be true whether solution strategies are retrieved whole-sale from a battery of potential strategies or if they are developed in-situ during the problem solving. It may be that during the execution of a problem solving strategy with more newly encountered tasks (i.e. when first learning to subtract three-digit numbers, or the first instances of mental division) students are devising new strategies during the actual problem solving process (Rathgeb-Schnierer & Green, 2013; Threlfall, 2009). These new strategies may be reliant on recognizing the numerical characteristics and arithmetic relations between the numbers presented in the problem. These in-situ creation of novel (to the individual) problem solving strategies may also occur after a set of procedures are already in place, when the situation and problem characteristics allows or requires it (e.g. Dowker, 1992). Even if an individual is simply choosing from a set of already established solution procedures they need to recognize the numerical characteristics and arithmetic relations embedded in the task in order to choose the most appropriate solution procedure for that task (Blöte, Klein, & Beishuizen, 2000; Canobi et al., 2003). It is therefore expected that no matter the nature of the problem solving, adaptive number knowledge supports the use of the most appropriate solution strategy for the individual in that situation.

Adaptive number knowledge has so far been explored most extensively in the form of arithmetic sentence production tasks (see McMullen et al., 2016 for more details). In the task, participants are asked to create as many correct arithmetic sentences that equal a target number (e.g. 16) from a set of five given numbers (e.g. 2, 4, 8, 12, 32) in 90 seconds. Those students with a more well-connected network of numerical characteristics and arithmetic relations are expected to be able to create more correct responses, along with responses that are more mathematically complex. Adaptive number knowledge has been found to be supported through explicit training with a serious game, the Number Navigation Game, in which students must constantly compare and mentally calculate multi-faceted arithmetic relations between numbers (Brezovszky et al., 2019)

Given that there are a number of possible global approaches to the arithmetic sentence production tasks, previous studies have aimed to examine the nature of inter-individual differences in adaptive number knowledge. These person-centered approaches have revealed substantial, and relevant, qualitative differences in the strategies and expression of adaptive number knowledge on the arithmetic sentence production task in primary school students (McMullen et al., 2016, 2017). Three groups with moderate performance, but different response patterns, were uncovered, which reveal important non-linear individual differences in adaptive number knowledge. The first of these three was a *Simple* group that performed at a slightly higher level than those in the more basic group, but who relied mostly on mathematically simple solutions that did not mix additive and multiplicative operations in a single arithmetic sentence. A second group was a *Complex* group that relied relatively heavily on mathematically complex solutions that used both additive and multiplicative operations in a single solution. Finally, the third group, the *Strategic* group, showed a particularly high level of adaptivity by matching their solution strategies to the item characteristics (e.g. Verschaffel et al., 2009).

Due to the data-driven approach of the previous analysis, it is possible that these results are idiosyncratic and would not appear in other samples or different age students. In particular, students' adaptive number knowledge may change as they learn more advanced mathematical content. More advanced skills and robust conceptual understanding of arithmetic that comes from learning algebra may change how some students respond on the arithmetic sentence production task. In order to examine this, the present study will investigate the structure of individual differences in adaptive number knowledge in a sample of lower secondary school students with at least one year of experience learning algebra.

Predictors of Adaptive Number Knowledge

Although a good deal is known about how adaptive number knowledge is related to other arithmetic knowledge and skills, little is known about the precursors of adaptive number knowledge. In particular, we aim in the present study to examine the relation between a number of relevant domain specific and general knowledge and skills and adaptive number knowledge.

Previous studies have revealed a relation between adaptive number knowledge and other arithmetic knowledge and skills. Among sixth graders, arithmetic fluency and arithmetic conceptual knowledge were both found to predict adaptive number knowledge, though the students' grades in mathematics class did not (McMullen et al., 2016). Furthermore, arithmetic fluency and arithmetic conceptual knowledge predicted adaptive number knowledge profile membership in the large scale study with fourth to sixth graders (McMullen et al., 2017). In particular there appeared to be differences in arithmetic fluency between all the profiles and differences between the lower and higher groups in arithmetic conceptual knowledge (i.e. there were no differences in conceptual knowledge between the three best-performing groups). However, it is possible that the inclusion of multiple grade levels in the same study may over-state the influence of fluency on adaptive number knowledge in this second study. As well, the three-step procedure used to test the relation between arithmetic fluency and conceptual knowledge and profile membership could not account for measurement error as well as a bias-adjusted three-step approach (Vermunt & Magidson, 2013), which takes into account classification uncertainty when examining the relation between class membership and covariates. The present study therefore investigates how adaptive number knowledge is related to arithmetic fluency within a single grade level (i.e., Grade 7) using a bias-adjusted three-step approach.

Arithmetic skill does not merely develop through explicit practice with arithmetic procedures during early, explicit instruction on these topics (Prather & Alibali, 2011). Improvements in arithmetic fluency may also occur with practice and development of more advanced, but related, mathematical topics, such as algebra. Individual differences in this later development – potentially from implicit instructional sources – may be related to adaptive number knowledge either (a) through higher adaptive number knowledge promoting the tendency to recognize numerical characteristics and arithmetic relations embedded in later mathematical instruction, or (b) by an increased encapsulation of arithmetic ties and procedures leading to improved adaptive number knowledge (Koponen, Salmi, Eklund, & Aro, 2013). In either case, increases in arithmetic fluency are not expected to uniformly impact adaptive number knowledge in later stages of arithmetic development. Instead, it is only at certain thresholds in development, such as starting to be able to use multiple operation solutions, that changes in arithmetic fluency would be expected to relate to adaptive number knowledge.

Adaptive number knowledge has been found to be uniquely related pre-algebra knowledge, even after taking into account arithmetic fluency and conceptual knowledge (McMullen et al., 2017). This suggests that adaptive number knowledge supports the development of the more advanced arithmetic reasoning that goes into pre-algebraic problem solving, especially involving missing-value problems. However, it is also possible that adaptive number knowledge may merely be a reflection of general mathematical achievement, as many features of mathematical cognition are often highly correlated with general mathematical achievement (Schenke, Rutherford, Lam, & Bailey, 2016). Thus, the present study aims to examine how general mathematical achievement is related to adaptive number knowledge with whole number arithmetic.

Adaptive number knowledge has proved to be a unique and well-founded feature of arithmetic knowledge and skills (McMullen et al., 2016, 2017). However, it has yet to be examined in relation to any domain general skills. Given the complexity of the instructions and novelty of the arithmetic sentence production task, which is unlike any task traditionally encountered in the mathematics classroom, it is possible that non-verbal intelligence may explain some variation in performance on the arithmetic sentence production task (Duncan, Schramm, Thompson, & Dumontheil, 2012).

Similarly, the arithmetic sentence production task may require students to rely on their working memory (Friso-van den Bos, van der Ven, Kroesbergen, & Van Luit, 2013). Working memory is a temporary memory system devoted to store and manipulate information (Baddeley, 1986). Working memory capacity predicts several basic academic abilities, such as mathematical skills and reading comprehension as well as more general fluid abilities in children and adolescents (e.g., Alloway & Alloway, 2010; Holmes & Adams, 2006; Seigneuric & Ehrlich, 2005). Individual differences in working memory capacity are seen as important predictor of arithmetic skills (Friso-van den Bos et al., 2013), irrespective of the nature of the processing component (Peng et al., 2016). The arithmetic sentence production task may place a particular burden on student's working memory as it requires them to keep in mind their target number and work forward and/or backwards in coming up with arithmetic sentence that equals that number. This burden may be particularly heavy in developing complex multi-operational solutions that use both additive and multiplicative operations, as these solutions may necessitate finding intermediate values in between the targets and given numbers, holding all of these in one's mind and negotiating the arithmetic relations between all of these. However, those students with a stronger basis of arithmetic fluency and conceptual knowledge may need to rely less on their working memory in coming up with multi-operational solutions.

The Present Study

In general, more evidence is needed to better situate adaptive number knowledge in the framework of adaptivity with arithmetic problem solving. Examining the nature and predictors of individual differences in adaptive number knowledge will better support our understanding of how adaptive number knowledge fits into the development of arithmetic skills. To achieve this end, the present study asks the following questions:

1. Do profiles of secondary students' adaptive number knowledge follow similar patterns as those profiles that were previously found among late primary school students?

Previously, both quantitative and qualitative differences in adaptive number knowledge were found among fourth to sixth graders (McMullen et al., 2017). Quantitative individual differences appeared in the overall number of correct responses. In addition, qualitative individual differences appeared in the response patterns and solution strategies used by students. Based on the previous latent profile analysis (LPA) of responses on the arithmetic sentence production task among 4-6 graders, the present study will use confirmatory LPA to examine whether similar profiles of adaptive number knowledge appear in lower secondary school students (end of Grade 7). Adaptive number knowledge has been shown to be related to later pre-algebra knowledge (McMullen et al., 2017). It is therefore possible that algebra instruction may have effects on the overall structure of adaptive number knowledge,

2. Do arithmetic fluency, the development of arithmetic fluency, general mathematical achievement, non-verbal intelligence, and working memory predict membership in profiles of adaptive number knowledge?

Previously, arithmetic skills and knowledge have been found to be related to adaptive number knowledge (e.g. McMullen et al., 2017). It is expected that arithmetic skills will be the strongest predictor of adaptive number knowledge profile membership, stronger arithmetic fluency leading to more advanced profiles. Change in arithmetic fluency is expected to be generally positively related to profile membership. It is expected that domain general cognitive components and general mathematical knowledge will also be positively related to adaptive number knowledge profile membership. A bias-adjusted step-three-step approach will be used to examine how these cognitive components are related to latent profiles of adaptive number knowledge.

Method

Participants

The present study is part of a broader longitudinal study (Ahonen & Kiuru, 2013-2017) that follows a community sample of Finnish students across the transition from primary school to lower secondary school (for more detailed description of the study and sample, please see Hirvonen, Väänänen, Aunola, Ahonen, & Kiuru, 2018; Mauno, Hirvonen, & Kiuru, 2018). The sample of this study consisted of 879 (473 girls, 54%) adolescents. A total of 841 adolescents participated in the research in the Fall semester of Grade 6 (fall 2014), and 838 adolescents participated in the Spring semester of Grade 6 (spring 2015). In the Spring semester of Grade 7 (spring 2016) there were 825 participants. In Grade 7 a total of 31 new adolescents joined the study (that did not participate in Grade 6). Parental written consent and child assent was required for student participation. The project was duly approved by the ethics committee of the last author's university.

Participants' ages ranged from 12 to 15 years (*M* = 12.3 years, *SD* = 4.36) in the Fall semester of Grade 6. In the first time point they were studying at 30 different schools, in 57 different classes, in large urban (80% of the participants) or mid-sized semi-rural (20%) towns in Central Finland. The participants' mother tongue was Finnish in 95% of cases, whereas 3% of the participants reported being bilingual with Finnish as a native language, and 2% of participants had a mother tongue other than Finnish. At the first time point most participants lived with both parents in one household (75%), or in turns with their mother and father (12%). 8% of participants lived only with their mother or father and the rest lived in blended or other types of families (5%). Out of the mothers 4% and out of the fathers 8% reported no vocational education after comprehensive school; 30% of mothers and 42% of fathers had completed lower secondary school, 40% of mothers and 29% of fathers had completed vocational college, and 26% of mothers and 21% of fathers had a Master's degree or higher. The sample was fairly representative of the Finnish general population (Official Statistics of Finland 2016a, 2016b).

Procedures

Students' data were collected during normal school days. All tests and questionnaires were group-administered by trained testers (there were always two trained research assistants present for each test situations). Students' non-verbal intelligence was tested in the Fall semester of Grade 6 and working memory in the Spring semester of Grade 6. Arithmetic fluency was tested both in the Fall semester of Grade 6 and in the Spring semester of Grade 7. Students' math performance and adaptive number knowledge were tested in the Spring semester of Grade 7.

Materials

Non-Verbal Intelligence (Grade 6, Fall Semester)

The Raven Standard Progressive Matrices (Raven, Raven, & Court, 1998) were used in assessing non-verbal intelligence. Raven's test consists of diagrams with one part missing. Students were asked to select the correct part that would complete each design, and the test increases in difficulty. In the present study, only half of the items were used and alternating items were selected to be presented (see also Kanerva et al., 2019). Responses were scored as correct or incorrect with the maximum score 30 (α = .81).

Working Memory (Grade 6, Spring Semester)

The Counting Span task was conducted using the touchscreen interface of an Android tablet (10.1 inches) with OpenSesame Runtime for Android (version 2.8.3) (Mathôt, Schreij, & Theeuwes, 2012; for more detailed description of the task and for the validity of the tablet version of the task, see Kanerva, Kiistala, Kalakoski, Hirvonen, Ahonen, & Kiuru, 2019): The assessment of children's working memory with a touch screen tablet in group settings: Validity and reliability. *Submitted manuscript*). The Counting Span task was a modified version of the task introduced originally by Case and colleagues (1982). Counting span task have been documented to predict arithmetic fluency as well as other scholastic skills in this age-group similarly to the reading span task (Kanerva et al., 2019). As a span task with increasing list length the counting span task is assumed to capture the WM capacity of adolescents of this age-group. The task presented sets of 3, 4, 5, 6, 7, or 8 yellow dots on a black tablet screen and consists of counting the number of dots (processing component) and storing the number of dots in each set in one's memory (storing component). After the presentation of the sets of dots, the participants were asked to entered the recalled number of dots in each set and input these answers by touching the numbers on the screen in the correct serial order. The task presented two sets of dots at first. After three trials, the amount of sets increased by one as long as the participant fails to recall all of the three sets. The partial credit unit scoring (PCU) (Conway et al., 2005) was adopted. In this scoring method, the mean number of correctly recalled memory items within a list length is calculated and these proportions are then averaged to obtain the score. This scoring method is recommended by Conway and colleagues (2005) based on solid internal consistency. Cronbach's alpha reliability for the Counting Span was .81.

Arithmetic Fluency (Grade 6, Fall Semester and Grade 7, Spring Semester)

Arithmetic Fluency was assessed with the Basic Arithmetic Test (Räsänen, Salminen, Wilson, Aunio, & Dehaene, 2009). The change in arithmetic fluency was based on the difference between students' scores from Grade 6 Fall to Grade 7 spring. The test contains tasks of addition, subtraction, multiplication and division. In this speeded test students were asked to do mental calculations and write their answers on the test paper. The test has 28 tasks (e.g., 527 + 31 = ?; 15 – ? = 9; 12 x 28 = ?), starting with easier ones and getting more difficult throughout the test. The participants were required to complete as many arithmetic operations as possible within a three-minute time limit. Responses were scored as correct or incorrect with the maximum score 28 (α = .82).

Mathematical Performance (Grade 7, Spring Semester)

Students' mathematical performance was assessed using a test for basic mathematical skills (Räsänen & Leino, 2005) designed for seventh to ninth graders (ages 12 to 15). The test has four versions from A to D, each comprising 40 items measuring different mathematical skills (e.g., basic calculation, story problems, and equations). The present study used version A. The test was performed by the students during a regular class with 18 minutes time limit. A maximum score in the test was 40 points, meaning that each item was worth one point. The Cronbach's alpha reliability for the mathematical performance was .84.

Adaptive Number Knowledge (Grade 7, Spring Semester)

Adaptive number knowledge was assessed using the arithmetic sentence production task (extensive details and analysis of this task can be found in McMullen et al., 2016). In the arithmetic sentence production task participants are asked to form as many mathematically correct arithmetic sentences that equal a target number using a set of five given numbers. Participants are told they are able to use the given numbers and the four arithmetic operations as many times as they want in any combination. Participants were given 90 seconds for each of the four test items described in Table 1. Reliability for the total number of correct answers across the four items was good (Cronbach's alpha = .82).

Table 1

Items of the Arithmetic Sentence Production Task During Pre- and Post-Test

Item	Given	Target	Type	Mean Correct	Max Correct
1	2, 4, 8, 12, 32	= 16	Dense	5.42	16
2	1, 2, 3, 5, 30	= 59	Sparse	1.98	8
3	2, 4, 6, 16, 24	= 12	Dense	5.85	16
4	2, 3, 6, 10, 18	= 38	Sparse	2.99	14

Two types of items are included in the arithmetic sentence production task (see McMullen et al., 2016 for a detailed explanation of item types): (a) dense items, Items 1 and 3, have numbers where there were a large amount of arithmetical relations that could easily be identified between the given and target numbers and (b) sparse items, Items 2 and 4, have a relatively small number of clear arithmetic relations between the given and target numbers.

For the LPA modeling, the participants' responses were separately coded as either *Simple* or *Complex*. Simple answers either only used additive operations (i.e. only addition and/or subtraction), or only used multiplicative operations (i.e. only multiplication and/or division). Complex answers contained both additive and multiplicative operations in the same solution (e.g. both addition and multiplication, 2*4+8=16). Sum scores for Simple and Complex responses were calculated for the dense items and sparse items separately.

Analysis

All analysis were pre-registered prior any analysis or detailed examination of the data; pre-registration protocol can be found at [masked for blind review]. In order to examine the profiles of adaptive number knowledge, LPA was used to model students' performance on the arithmetic sentence production task using four indicators of the number of correct simple responses on dense items (Dense Simple), complex responses on dense items (Dense Complex), simple responses on sparse items (Sparse Simple), and complex responses on sparse items (Sparse Complex). The normal procedure of modelling an LPA in which a one-class solution is estimated and the number of classes in the model is increased one class at a time. As well, since we aimed to examine if the LPA model used to describe 4^{th} to 6^{th} graders' performance on the arithmetic sentence production task could be also found in the present sample, we also constructed a confirmatory LPA model of this model. This was estimated using relative constraints for each of the four indicators using the findings from the previously identified model, in which there were five classes that aligned as: Basic, Simple, Complex, Strategic, and High. Since these four values capture all the solutions students provided, total score is not included as an indicator in the LPA model. The following constraints were placed on the confirmatory model: Dense Simple: Basic < Simple = Complex = Strategic = High; Dense Complex: Basic = Simple = Strategic < Complex < High; Sparse Simple: Basic = Complex = Strategic < High < Simple; Sparse Complex: Basic = Simple < Complex = Strategic < High.

LPA modelling was conducted with Mplus version 7.4 (Muthén & Muthén, 1998-2017) maximum likelihood with robust standard errors was the estimation method, with 500 and 50 random starts were employed to avoid local dependences. Models were statistically evaluated by Bayesian Information Criterion (BIC) and sample size adjusted BIC (aBIC) minima and significant results of the Lo–Mendell–Rubin (LMR) test, and the bootstrap likelihood ratio test (BLRT) indicating that the k profile model is preferred over the k-1 profile model. Entropy for the most appropriate model should exceed .8 (Lanza, Collins, Lemmon, & Schafer, 2007).

A three-step approach will be taken to identify the relation between adaptive number knowledge profiles and the cognitive and mathematical covariates, this allows for the determination of the latent classes without influence from the covariates. The three-step approach examines the relation between external variables and profile membership only after the LPA model has been identified (as a *third* step), thus it does not have any influence on the profile structure nor profile membership. The bias-adjusted three-step model used in the present study also takes into account uncertainty and error in assigning individuals to their most-likely profiles (Vermunt & Magidson, 2013).

Results

Table 2 reports the descriptive statistics for each of the four indicator variables used in the LPA modelling and the five covariates.

Table 2

Descriptive Statistics for Indicator and Predictor Variables

Variable	*M*	*SD*	Skewness (*SE*)	Kurtosis (*SE*)	Range
Dense Simple	10.21	3.92	0.41 (.09)	0.48 (.17)	0 − 29
Dense Complex	1.06	1.45	1.80 (.09)	3.71 (.17)	0 − 9
Sparse Simple	1.90	1.76	1.05 (.09)	1.15 (.17)	0 − 10
Sparse Complex	3.08	2.20	1.08 (.09)	2.37 (.17)	0 − 15
Arithmetic Fluency (Grade 7)	15.46	3.54	-0.54 (.09)	0.97 (.17)	0 − 26
Arithmetic Change	1.85	2.60	-0.02 (.09)	0.29 (.18)	-7 − 11
Math Achievement	16.97	5.64	-0.28 (.09)	-0.01 (.17)	0 − 33
Non-verbal Intelligence	22.63	3.67	-1.18 (.08)	2.31 (.17)	5 − 29
Working Memory	6.97	3.51	0.28 (.09)	-0.32 (.17)	0 − 19

LPA of Adaptive Number Knowledge

Table 3 details the statistical indicators for the three to six class solutions for the LPA, along with the 5-class confirmatory model. As can be seen, BIC and aBIC minima were found for the six class model (the seven class model did not reliably converge even with random starts increased to 5000 and 500). However, the six class model contained two classes with less than 5% of the sample, both of which had the same profile structure (high performance on all indicators). Basically, the six class model was structurally similar to the five class model and the additional class only further differentiated between the very top performers with very little added value. Additionally, the BLRT was not informative and the LMR test suggested the four class model to be the most appropriate. Taking all indicators and theory into consideration, we argue that the non-confirmatory five class model may be the most parsimonious way to represent this data in this sample.

Table 3

Statistical Indicators for LPA Modelling

Number of Classes	BIC	aBIC	Entropy	BLRT	LMR
3	8628	8571	.91	.0000	.02
4	8515	8442	.86	.0000	.003
5 confirmatory	8467	8407	.77	.500	NA
5	**8451**	**8362**	**.83**	**.0000**	**.20**
6	8384	8279	.86	.0000	.60

Note. Bold text indicates the selected model.

The mean values for each indicator by latent profile in the five class solution can be found in Table 4, along with the overall distribution of the sample into the different classes. Even the Basic class provided an average of around thirteen correct responses across all four items, the bulk of which are simple solutions on the dense items. The Simple class stood out in their use of simple solutions on the sparse items. These solutions avoid the more mathematical complex relations on these items (e.g. using repeated addition instead of multiplication and addition). The Complex class appears to be rely more on complex solutions on the dense and sparse items; they have a fairly average amount of simple solutions. Interestingly, along with the High class, the Complex class is the only class to provide a substantial number of complex solutions on the dense items, when these are less needed due to a large number of more straightforward relations between the given and target numbers. In contrast, the Strategic class seems to shift strategies between the dense and sparse items, producing a relatively large number of simple solutions on the dense items, where they are numerous, and complex solutions on the sparse items, where they are more necessary. The High group appears to be especially apt at producing complex solutions on all items, even while producing a decent amount of simple solutions.

Table 4

Means for All Four Indicators by Latent Profile. Percent of Sample Based on Estimated Posterior Probabilities

Profile	Dense Simple	Dense Complex	Sparse Simple	Sparse Complex	Total Correct	Percent of Sample
Basic	8.45	0.37	1.33	2.17	12.32	58.8
Simple	13.99	0.64	4.94	2.32	21.89	10.2
Complex	11.32	2.61	1.96	4.37	20.26	19.6
Strategic	14.65	0.56	1.71	5.99	22.91	7.4
High	12.68	5.63	2.46	6.53	27.30	4.0

In general, the five class solution showed a large amount of similarity with the previous LPA of adaptive number knowledge. Figure 1 shows the two five class models side-by-side for a direct comparison. Although the overall structures of the profiles are similar, there are two main differences: (a) the overall levels of the scores for the Simple, Strategic, and High profiles and (b) the class membership distributions. Among the present sample of 7th graders, the Basic, Simple, and High classes appear larger and the Strategic class smaller, while the Complex class remains relatively large. In all, it appears that even though the confirmatory model was not the most appropriate in this sample, the results of the previous LPA of adaptive number knowledge were fairly well replicated.

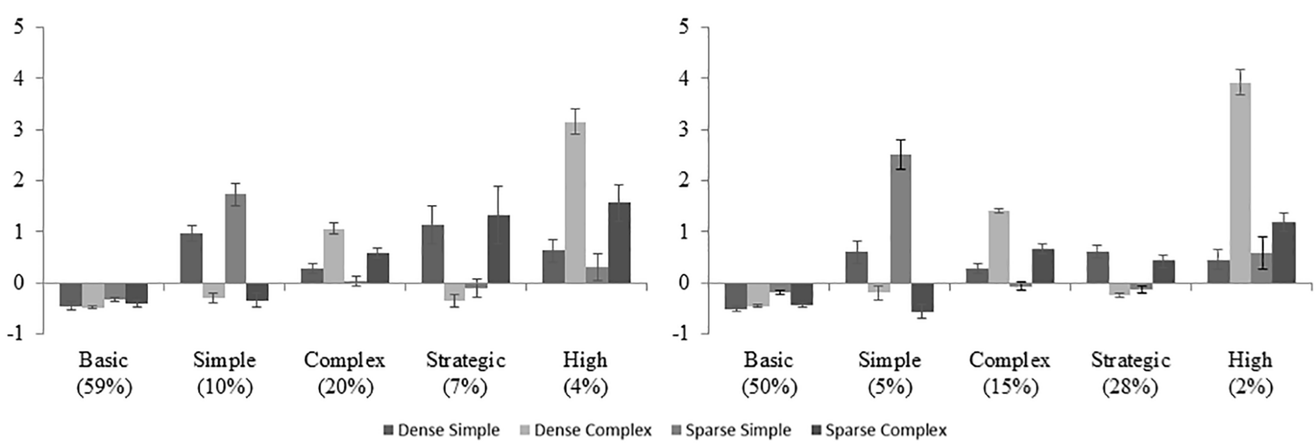

Figure 1. Standardized mean values for each latent profile for the present study involving 7th graders on the left and the study by McMullen and colleagues (2017) involving 4th to 6th graders on right. Error bars represent +/- 1 standard error.

Predictors of Adaptive Number Knowledge

The three-step approach provided an examination of the differences between the adaptive number knowledge profiles on a variety of cognitive and mathematical measures. All covariates were added simultaneously and revealed that there were differences across the profiles in arithmetic skills, the development of arithmetic skills, and general mathematical achievement. Figure 2 shows the results of the multinomial logistic regression with all five covariates included with the Basic class as the reference class.

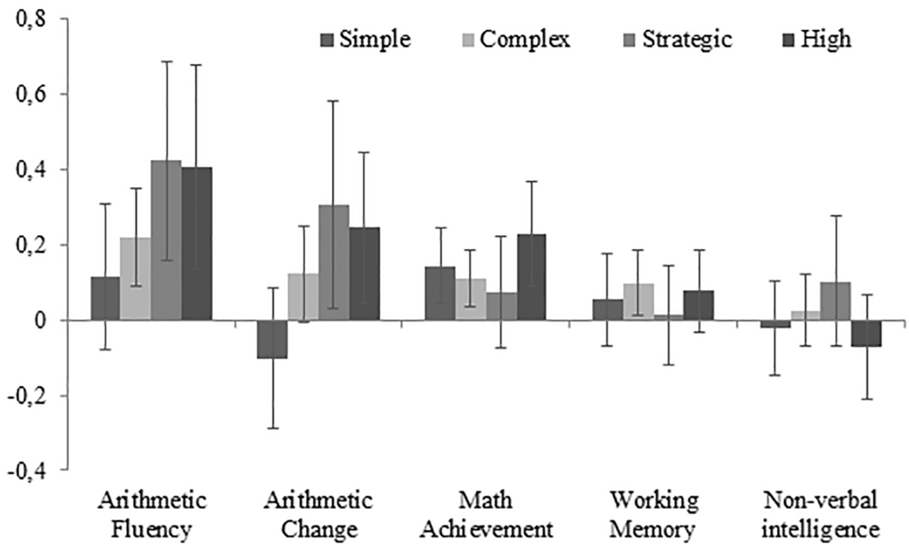

Figure 2. Multinomial logistic regression coefficients based on the three-step procedure. Basic class is the reference class and all values represent profile derivations from Basic class values (set at 0). Error bars indicate +/- 2 standard error of mean difference between Basic class and each class (and cannot be used to compare other classes with each other, e.g. Strategic versus High).

In particular, two distinct patterns emerged with regard to arithmetic knowledge. First, the Basic profile had lower arithmetic fluency than the Complex, Strategic, and High profiles. Second, the Basic profile had a smaller changes in arithmetic than the Strategic and High profiles, and the Simple profile had smaller changes in arithmetic than the Complex, Strategic, and High profiles. As well the Basic profile had lower mathematical achievement than the Simple, Complex, and High profiles. There were no statistically significant differences between the profiles in non-verbal intelligence nor working memory (except for a small difference between the Complex and Basic profiles for working memory). In all, one striking finding is that for these five predictors there were no significant differences between the three groups with the strong adaptive number knowledge, namely, the Complex, Strategic, and High profiles.

Discussion

Adaptive number knowledge has been identified as a feature of arithmetic development and described as an underlying feature of adaptivity with arithmetic problem solving alongside procedural flexibility (McMullen et al., 2016, 2017). The present study provides further evidence of the unique nature of adaptive number knowledge in arithmetic development in two ways. First, the structure of individual differences in adaptive number knowledge appeared to be the same in seventh graders as in primary school students. This indicates that indeed there are particular manifestations of adaptive number knowledge that varied between individuals in reliable manners, both during explicit instruction with arithmetic and after these topics have been exhaustively covered in the mathematics curriculum and more advanced topics, especially algebra, have been introduced. Second, adaptive number knowledge was found to be uniquely related to procedural fluency with arithmetic and its development even after taking into account general mathematical achievement and domain general cognitive skills. However, adaptive number knowledge was not entirely explained by procedural fluency with arithmetic, as the most advanced adaptive number knowledge profiles did not differ in arithmetic fluency or its development.

Profiles of Adaptive Number Knowledge

The profile structures of adaptive number knowledge in seventh graders in the present study were remarkably similar as was found previously in fourth to sixth graders (McMullen et al., 2017). Considering that by the end of seventh grade Finnish students have had a year of instruction in lower secondary school mathematics, including algebra, it is surprising that the structures of these students' response patterns for the arithmetic sentence production task are so strikingly similar with the fourth to sixth graders, many of whom had not completed whole number arithmetic instruction.

In both the present study and in the previous study, the Basic profile was the most prominent, with the majority of students in both samples (50% among 4-6 graders and 59% among 7 graders in this study). While this profile is labeled as Basic, the mean scores suggest some success in coming up with correct solutions, even the most advanced type of solutions, complex solutions on sparse items. This result suggests that adaptive number knowledge may be most differentiated at the top-levels of arithmetic performance. That the Basic performance level improves between fourth and seventh grade is not surprising. What is surprising is that more students would be placed in this category among older students. This suggests a widening gap in the development of adaptive number knowledge across grade levels, with fewer students exceeding their peers in exceptional ways. Increasing disaffect towards math in lower secondary school might be an issue at play, especially given the open-ended nature of the task (e.g. find as many solutions as you can). Longitudinal investigation into the development of adaptive number knowledge would be beneficial for better answering these questions.

Despite the large(r) number of students in the Basic profile in the present study, there still appears to be substantial and meaningful differentiation in adaptive number knowledge with these secondary school students at the upper levels of this knowledge. In particular the Simple, Complex, Strategic, and High profiles all have ostensibly high levels of success on the arithmetic sentence production task in terms of overall scores. But, they do so in with different solution patterns. In particular, there seems to be a difference in the response patterns for the Complex and Strategic profiles on the Dense items. In general, students did not use complex solutions on the dense items; presumably this is because it is not necessary to do so, given the large number of both simple and more complex arithmetic relations between the given and target numbers. While the Strategic profile appeared to vary their strategies depending on the item type, the Complex profile was exceptional in their use of complex solutions on the dense items (outside of the High profile).

Potentially, these patterns for producing answers on the task are a conscious strategy, though this also may be a reflection of the open nature of the task. Both groups were just as likely to produce complex solutions on the Sparse items, however. The Strategic profile even had a higher number of complex solutions on these items than the Complex profile. This suggests that the Strategic profile may more reflect a group of students with a higher general flexibility in solution strategies, who is able to adapt their solution strategy to fit the problem type. More details on the actual solutions provided by these groups may shed light on the nature of these differences and provide more insight into how adaptive number knowledge may be promoted in the classroom.

Predictors of Adaptive Number Knowledge Profile Membership

The present study provides more evidence that adaptive number knowledge is a distinct component of arithmetic skills and knowledge. Arithmetic fluency and change in arithmetic fluency were found to predict profile membership even after taking into account general mathematical achievement and domain general cognitive abilities. However, there appeared to be substantial individual differences in adaptive number knowledge that were not explained by these mathematical and other cognitive skills and knowledge.

Crucially, the present study was able to include a measure of the change in arithmetic fluency from the beginning of Grade 6 until the end of Grade 7, when arithmetic procedures are not any longer explicitly taught in the classroom. Improvements in arithmetic fluency during this period of non-explicit instruction may be representative of increased automatization of basic numerical facts (Koponen et al., 2013), which could support more complex arithmetic processes, such as reasoning on the arithmetic sentence production task. Arithmetic fluency has been previously found to be strongly related to profiles of adaptive number knowledge in younger students (McMullen et al., 2017), these results were partially replicated in the present sample. The connection between procedural fluency and flexibility has been identified in other domains in mathematics (Schneider, Rittle-Johnson, & Star, 2011), and these results further confirm this.

However, it appears that performance on the arithmetic sentence production task is not related to general nonverbal intelligence, after taking into account more specific mathematical skills. Nor does working memory seem to distinguish between profiles of adaptive number knowledge. It is possible that the effects of these domain general abilities are already embedded in individual differences in arithmetic and mathematical skills and knowledge and therefore do not provide any unique explanation for differences in adaptive number knowledge (Alloway & Passolunghi, 2011). As well, the arithmetic sentence production task requires retrieval of previously stored, well-learned arithmetical operations. Although working memory has strong relationship with broad measures of mathematical attainment, it has been reported that it has a weaker relationship with arithmetic fluency (Fuchs et al., 2005). Thus, it is possible, that the ability to store and process information in working memory is not that crucial for the arithmetic sentence production task, and the ability to efficiently retrieve arithmetic and numerical information from long-term memory may be more important. This conclusion is further supported by the numerical content of the working memory task, which could potentially overstate the relation between counting span and adaptive number knowledge. However, complex span tasks appear to assess a general cognitive capacity and predict mathematical skills, as well as other complex cognitive skills, irrespective of the domain of the WM task (Peng et al., 2018).

The notion of adaptive number knowledge as high-level knowledge is supported by the lack of differences between the Complex, Strategic, and High profiles for any of the covariates included in the study. Thus, adaptive number knowledge may be the main cause for individual differences in performance on the arithmetic sentence production task at these upper levels. Previously, in the sample of fourth to sixth graders these same three top-level profiles were found to significantly differ in arithmetic fluency (McMullen et al., 2017). However, these differences may have been partially overstated due to the inclusion of multiple grade levels in the same model. In the present study, when only one grade level is included, there do not appear to be any differences in arithmetic fluency among these profiles.

The present study is the first to present evidence that students' development of arithmetic fluency positively predicted adaptive number knowledge. The question that arises, given that this was not a causal relation, was whether this change led to better adaptive number knowledge or whether better adaptive number knowledge led to more improved fluency. A third possibility is that both of these directional relations are true, with an positive iterative loop being the cause of this relation (e.g. Hannula & Lehtinen, 2005). It is possible that those students with higher adaptive number knowledge would be better able to recognize and internalize (encapsulate) the numerical characteristics and arithmetic relations embedded in more advanced mathematical tasks, leading to improvements in their arithmetic fluency. These improvements in arithmetic fluency would then support the further development of adaptive number knowledge.

In the present study one of the biggest clarifying correlations is the differences between the Simple profile and more advanced profiles in arithmetic fluency and change in arithmetic fluency. While there were no differences between the Simple profile and the more advanced profiles in their arithmetic fluency in Grade 6, the Simple profile had significantly less development of arithmetic fluency over the next two school years than the three advanced groups. This suggests that the Simple group do not seem to draw out the key aspects of arithmetic implicitly from more advanced mathematical topics such as algebra, which would support their development of arithmetic fluency in Grades 6 and 7.

Limitations and Conclusions

There are a number of limitations to the present study which should be addressed in future studies on adaptive number knowledge. The first being that the present study would be better suited to include more diverse measures of adaptive number knowledge. So far, the arithmetic sentence production task is the only task that has been used in assessing adaptive number knowledge, and it is not clear that this is a more generalizable type of knowledge. More measures which tap into assessing students' knowledge of numerical characteristics and arithmetic relations in novel tasks would be valuable for determining more exactly the nature of adaptive number knowledge. Relatedly, the present study only included arithmetic measures of procedural fluency. Future studies should aim to assess how conceptual knowledge, procedural flexibility, and general adaptivity with arithmetic problem solving are related to adaptive number knowledge. Including a more comprehensive battery of arithmetic measures would better situate adaptive number knowledge in this domain.

Despite these limitations, the present study provides a strong step forward in deepening our understanding of the nature of adaptive number knowledge and its role in the development of arithmetic. As more and more mathematical curricula turn their focus to adaptable skills and knowledge that fall under the guise of adaptive expertise, finding foundational skills and knowledge that would support such goals is crucial (Baroody, 2003; Mullis, Martin, Goh, & Cotter, 2016). Adaptive number knowledge, as it is presented here, and in previous studies, provides a new approach to examining and assessing arithmetic skills on a more global level. Flexibility and adaptivity with arithmetic problem solving has gained a good deal of attention in the mathematical education and educational psychology domains in the previous years (Torbeyns, Ghesquière, & Verschaffel, 2009; Verschaffel et al., 2009), yet little has been accomplished in developing broad approaches to what these skills and knowledge looks like. While more evidence is needed in order to clarify the nature of adaptive number knowledge, this conceptual replication of the nature of individual differences in adaptive number knowledge is a strong step towards such clarity.

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Competing Interests

The authors have declared that no competing interests exist.

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Data Availability

Due to the on-going nature of this research project, which is a part of a larger longitudinal study, the STAIRWAY-From Primary School to Secondary School Study, we are not able to make the data publicly available at the moment. The metadata information is, however, freely available at University of Jyväskylä. Some years after finishing all longitudinal data collections we will archive non-sensitive parts of the data in the anonymous format in Finnish Social Science data Archive, which will be directly accessible via the project page at https://osf.io/469va/

Supplementary Materials

The supplementary material contains the Mplus input file (for access, see Index of Supplementary Materials below).

Index of Supplementary Materials

McMullen, J., Kanerva, K., Lehtinen, E., Hannula-Sormunen, M. M., & Kiuru, N. (2019). *Supplementary materials to "Adaptive number knowledge in secondary school students: Profiles and antecedents".* PsychOpen. https://doi.org/10.23668/psycharchives.2670

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