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The Power of One: The Importance of Flexible Understanding of an Identity Element

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Supplementary Materials: Materials [see Index of Supplementary Materials]

Abstract

The number one plays a special role in mathematics because it is the identity element in multiplication and division. The present findings, however, indicate that many middle school students do not demonstrate mathematical flexibility representing one as a fraction. Despite possessing explicit knowledge of fraction forms of one (e.g., 95% of students indicated that 36/36 = 1), most students did not recognize and apply knowledge of fraction forms of one to estimate numerical magnitudes, solve arithmetic problems, and evaluate arithmetic operations. Specifically, students were less accurate in locating fraction forms of one on number lines than integer forms of the same number; they also were slower and less accurate on fraction arithmetic problems that included one as a fraction (e.g., 6/6 + 1/3) than one as an integer (e.g., 1 + 1/3); and they were less accurate evaluating statements involving fraction forms of one than the integer one (e.g., lower accuracy on true or false statements such as 5/6 × 2/2 = 5/6 than 4/9 × 1 = 4/9). Analyses of three widely used textbook series revealed almost no text linking fractions in the form n/n to the integer one. Greater emphasis on flexible understanding of fractions equivalent to one in textbooks and instruction might promote greater understanding of rational number mathematics more generally.

Keywords

fractions, fraction arithmetic, mathematical flexibility, textbook analysis, flexible understanding of number

The number one plays a key role in rational number arithmetic. It is the identity element in multiplication and division. It allows for the generation of equivalent fractions (e.g., 4/5 × 2/2 = 8/10). Moreover, fraction forms of one are used in many standard mathematical procedures, as when adding and subtracting fractions with unequal denominators (Table 1).

Table 1

Examples of Uses of the Number One

Context	Use
Fraction addition/ subtraction	Adding and subtracting fractions with unequal denominators requires multiplying by one or n/n (e.g., 2/3 + 3/4 = (2/3 × 4/4) + (3/4 × 3/3) = 8/12 + 9/12 = 17/12).
Fraction Division	A justification that is based on multiplying both numerator (a/b) and denominator (c/d) by the reciprocal of the denominator (d/c): that is, a/b ÷ c/d = (a/b × d/c) / (c/d × d/c) = (a/b × d/c) / 1 = (a/b × d/c).
Arithmetic Shortcuts	Knowing that any number divided by itself is one allows for arithmetic shortcuts (e.g., 49/79 × 977/977 = 49/79 × 1 = 49/79).

Extending knowledge of the integer one to fraction forms of one (e.g., 2/2, .5/.5, 48/48, etc.) is a component of *mathematical flexibility,* which involves recognizing as well as applying knowledge of the relations among numbers and varied strategies to arrive at a solution (McMullen et al., 2016, 2017; Star et al., 2015). Such flexibility is critical for a wide variety of mathematical outcomes. For example, flexibly using knowledge of the relations among numbers and arithmetic procedures is predictive of later algebra knowledge, more predictive than routine use (McMullen et al., 2017, 2020). Recognition of the value of mathematical flexibility led the authors of the Common Core State Standards (2010) to make it a major goal of mathematics education.

Flexible application of knowledge requires not only willingness to be flexible but also the knowledge that can be applied flexibility. For example, if students lack knowledge that multiplying or dividing a number by any version of one leaves unchanged the value of the number being multiplied or divided by one, they cannot apply that knowledge to solve problems, regardless of their inclination to be flexible. Much of the focus of the present research is on whether children possess the knowledge required to flexibly apply one in different forms to solve problems.

Fully understanding the standard procedure for adding fractions with unequal denominators requires the knowledge that multiplying by n/n leaves unchanged the value of the number being multiplied. Although it is possible to generate equivalent fractions by only focusing on the independent whole number components (i.e., simply multiplying the numerator and denominator by the same number), lack of understanding that multiplying by n/n is the same as multiplying by the integer one may reinforce misconceptions that multiplication makes the equivalent fraction "bigger" (Muzheve & Capraro, 2012). Without flexible understanding of one as a fraction, students also struggle to understand a common justification of the invert-and-multiply procedure for division of fractions—that multiplying both divisor and dividend by the inverse of the dividend leaves unchanged the value of the original problem. Moreover, understanding the role of the number one in the generation of equivalent fractions could aid in later algebra understanding. For example, Yakes and Star (2011) suggest that understanding the identity element aids in simplifying algebraic expressions when it is not immediately clear what common number the numerator and denominator must be divided by (e.g., $\frac{2a^2b}{6ab^3}$).

Understanding the equivalence of different forms of one makes possible flexible application of that knowledge. From an adaptive strategy choice framework (Siegler, 1996), solving mathematics problems requires choosing among three types of strategies: correct and efficient, correct but inefficient, and incorrect. For example, if presented 57 1/4 − 56 3/4, a correct efficient strategy would be to convert 1 whole from 57 and combine it with 1/4 before subtracting 3/4 (i.e., 56 5/4 − 56 3/4). A correct inefficient strategy would be to convert both whole numbers to improper fractions, yielding 229/4 − 227/4. An incorrect strategy would be to subtract the smaller from the larger whole number and the smaller from the larger fraction, yielding 1 2/4. The correct efficient strategy is based on 4/4 and 1 being equivalent; that strategy would be faster than the correct inefficient strategy and reduce multiplication errors in the conversion process. Thus, flexible application of knowledge of the number one is likely to facilitate speed and accuracy for many aspects of fraction arithmetic, among other advantages.

In the next section, we discuss current knowledge about children's understanding of varied forms of one. Then, we examine the skill with which children flexibly use the number one to solve problems (Study 1). Finally, we analyze a potential contributor to the limited understanding of one that children displayed in Study 1 − that textbook problems rarely address varied forms and uses of one (Study 2).

Children's Understanding of the Number One

At present, little is known about flexible understanding of one. A study that was informative was a teaching experiment by Mack (1995) that indicated that children had difficulty connecting symbolic and non-symbolic fraction forms of one. The study provided interesting qualitative descriptions, but lack of quantitative analyses rendered unclear the prevalence of such difficulties. Another relevant study was McMullen et al. (2020), which showed that children less frequently generated equations involving fraction forms of one (e.g., 4/4 = 1) than other types of equations when asked to generate as many equations equal to one as possible in a limited time from a set of four numbers. It was unclear, however, whether this disparity reflected lack of flexible understanding of one or whether participants simply preferred other operations for generating such equations.

Classroom instruction regarding one as a fraction sometimes includes incorrect statements, which could undermine children's understanding of fraction forms of one. In a classroom observation study of spoken and written instruction of 16 teachers, errors of both omission and commission were present in lessons involving fraction forms of one (Muzheve & Capraro, 2012). Some teachers substituted exponents when they meant to convey multiplication by N/N (e.g., $\frac{4^2}{5^2} = \frac{8}{10}$). Other teachers wrote symbols on the board that indicated that they were multiplying by a whole number (e.g., $\frac{4}{5} \times 2 = \frac{8}{10}$) instead of by a fraction equivalent to one (e.g., $\frac{4}{5} \times \frac{2}{2} = \frac{8}{10}$). Similar written inaccuracies later appeared in students' work in those classrooms (up to 76% of students per class exhibited at least one of these issues). The students often answered correctly despite their work including such false statements, but the incorrect statements may have adversely affected their understanding of why the mathematical procedures make sense.

Study 1

In Study 1, we examined children's flexible understanding of the number one in the contexts of solving problems involving fraction magnitude representations and fraction arithmetic and answering explicit questions about both. Classroom observations of many children's lack of understanding of the identity element stated as a fraction (e.g., Mack, 1995) and research documenting children's limited generation of fractions equivalent to one to solve problems requiring flexible application of this knowledge (McMullen et al., 2020) suggested that children would demonstrate weak knowledge of one in fraction form (n/n). On the other hand, textbooks often state as a rule that n/n equals one. For example, the 4th grade volume of *Everyday Math* states that "n/n means n out of n equal parts. That's all of the parts of 1 whole" (p. 127) and "when the numerator and denominator are the same, the fraction is equal to one" (p. 153) (University of Chicago School Mathematics Project [UCSMP], 2015).
Therefore, we predicted:

1. Students possess factual knowledge that fraction forms of one equal one; if asked whether n/n = 1, they will know that it does.
2. Students will answer slower and less accurately on arithmetic problems involving fractions equivalent to one (e.g., 6/6 + 1/3) than on equivalent problems involving one as an integer (e.g., 1 + 1/3).
3. Estimates of fraction forms of one on number lines will be less accurate than estimates of the integer form of one.
4. Students' lack of flexible understanding will extend to answering explicit questions regarding one as a fraction that require conceptual understanding.

Method

Participants were 49 middle school students (45% female, 10 6th graders, 15 7th graders, and 24 8th graders), attending two suburban schools in different states in the Northeastern U.S (the sample was approximately evenly split between the two schools). In one school district, 14% of students received free or reduced-price lunch, as compared to the state average of 38%. In the other district, 28% of students received free or reduced-price lunch, as compared to the state average of 47%. Middle school students were chosen, because they would have completed all or almost all instruction in fraction arithmetic (Common Core State Standards Initiative [CCSSI], 2010). The Teachers College, Columbia University Institutional Review Board approved the study.

Tasks

The three tasks – fraction arithmetic, fraction number line estimation, and explicit questions about fraction arithmetic – were presented online. All problems appear in the Supplementary Materials.

Fraction Arithmetic — Children were presented four types of fraction arithmetic problems. Three included an operand equivalent to one: an integer (e.g., 5/6 × 1); a one-digit fraction (e.g., 5/6 × 8/8); or a two-digit fraction (e.g., 5/6 × 54/54). On the fourth type of problem, neither operand equaled one and both operands were one-digit fractions (e.g., 5/6 × 3/8). Four items of each type were presented (one for each arithmetic operation) for a total of 16 problems. Time spent on each item was recorded to indicate the effects of integer and fraction forms of one on solution times, relative to each other and to problems where neither operand equaled one.

Number Line Estimation — Children were asked to estimate the magnitudes of individual fractions on a computer screen by moving the cursor to the desired position on the number line and clicking there. Children first estimated the magnitudes of 10 fractions (1/19, 2/13, 1/5, 1/3, 3/7, 7/12, 5/8, 3/4, 7/8, and 13/14) on a 0-1 number line. This allowed direct comparison of their performance with that of children in Siegler and Pyke (2013), which illustrated general difficulties that students have understanding fraction magnitudes.

We also examined magnitude understanding for different forms of one. On these 18 trials, children were asked to estimate magnitudes on 0-1, 0-2, and 0-100 number lines of numbers that corresponded to points 25%, 50%, and 75% of the distance between the endpoints, and also to estimate the magnitude of the number one in integer, one-digit fraction, and two-digit fraction forms on each line. The fraction 50/50 was included for each numerical range, to examine effects of numerical range on estimates of a number that was identical in both magnitude and surface structure. On the 0-1 line, students estimated 2/4, 30/40, 1, 47/47, 3/3, and 50/50; on the 0-2 number line, they estimated 1, 3/2, 4/2, 89/89, 2/2, 30/40, and 50/50; on the 0-100 line, they estimated 100/2, 75/1, 200/2, 1, 36/36, 9/9, and 50/50. All three number lines were the same physical length.

Questions Assessing Explicit Understanding — To probe specific understandings and misunderstandings when computational requirements were not competing for working memory resources, children were presented 16 true/false and multiple-choice questions assessing explicit understanding of one. With these questions, we aimed to understand whether children had explicit knowledge of fractions equal to one (Category 1), whether children performed better on problems that were identical (Category 2) or equivalent but not identical (Category 3) to the literal answer yielded by the multiplication procedure, and whether children understood the connection between fractions equivalent to one and mixed numbers (Category 4). Categories 1 and 2 require knowledge of fraction forms of one but not flexible application of the knowledge; Categories 3 and 4 require both the knowledge and its flexible application. Table 2 displays examples of items in each category.

After each true/false question, children were asked to rate their confidence that their answer was correct. Incorrect procedures and inaccurate answers can reflect either a belief that such procedures are correct, in which case confidence ratings would be high, or a “could be” attitude, in which case confidence would be low. Past research (Fitzsimmons et al., 2020; Siegler & Pyke, 2013) has indicated that accuracy and confidence reflect different processes.

Students rated their confidence by moving a slider along a number line, with the left end labeled “Not Confident” and the right end labeled “Very Confident.” For purposes of analyses, the number line was divided into 100 sectors (not visible to the children); the point to which children moved the slider indicated their confidence in their answer on a 0-100 scale. Students could not change their previous answer during confidence assessment questions. Due to experimental error, confidence ratings were only obtained on the true/false questions.

Table 2

Four Categories of Explicit Questions and Accuracy on Each

Category	Category Type	Example	Percent Correct	*SD*
1	Knowledge of Individual Fractions	T/F: 36/36 = 1	86	20
2	Cases Where Fraction Multiplication Procedure Yields Literal Answer	T/F: 3/4 × 2/2 = 6/8	62	23
3	Cases Where Fraction Multiplication Procedure Yields Equivalent Answer	T/F: 2/3 × 1/2 = 1/3	33	32
4	Relations Between Fractions and Mixed Numbers	1 + 2/3 ☐ 5/3	24	31

Procedure

Problems were presented on an online survey platform, Qualtrics, during a remote math class period (due to the COVID-19 pandemic). Tasks were presented in a fixed order: first, fraction arithmetic; then, number line estimation; then, explicit knowledge questions. Order of presentation of problems was randomized within each task.

Results

Bonferroni post-hoc tests ($p < .01$) were used to determine the locus of all interactions and all main effects in which a variable had three or more values. Only statistically significant differences were reported. The Huynh-Feldt correction was reported whenever Mauchly's test indicated there was a violation of the assumption of sphericity.

Prediction 1: Students possess factual knowledge that fraction forms of one equal one; if asked whether n/n = 1, they will know that it does.

Students were highly accurate (*M* = 95%) and confident (84%, *SD* = 25%) at indicating 36/36 = 1.

Prediction 2: Students will answer slower and less accurately on arithmetic problems including fractions equal to one than on equivalent problems including one as an integer.

A one-way repeated measures ANOVA on accuracy on the four types of arithmetic problems revealed the predicted main effect of operand type, $F(3, 138) = 17.933$, $p < .001$, $\eta_p^2 = .280$. Problems with the integer one as an operand were solved more accurately (62.23% correct, *SD* = 30.78%) than items with a one-digit fraction form of one (46.27% correct; *SD* = 31.03%, $p = .001$), a two-digit fraction form of one (33.87% correct, *SD* = 35.5%, $p < .001$), or no form of one (39.01% correct, *SD* = 35.62%, $p < .001$). Problems with fraction forms of one were solved no more often than problems without an operand of one.

A parallel one-way repeated measures ANOVA on response times for the four types of problems indicated that operand type also influenced response times, $F(2.26, 103.85) = 11.5$, $p < .001$, $\eta_p^2 = .20$. Participants were faster on problems with an integer form of one (24.3 s, *SD* = 19.67 s, $p < .001$) than on problems with a one-digit fraction form of one (54.5 s, *SD* = 38.2 s), a two-digit fraction form of one (64.4 s; *SD* = 60 s), or no form of one (51.8 s, *SD* = 31.9 s). No differences in response times were present on problem types where both operands were fractions. These data again suggested that participants used their knowledge of one when it was presented as an integer but not when it was presented as a fraction.

An analysis of speed and accuracy on individual trials supported this conclusion. If children applied knowledge of one to multiplication and division problems, such as 5/6 × 1 and 1/2 ÷ 1, they should be both fast and accurate on them. To generate a quantitative estimate of how frequently children applied knowledge of varied forms of one on multiplication and division problems, we coded each problem for whether the answer was both correct and relatively fast. Answers of 10 seconds or less were defined as fast; the intent of this relatively generous criterion was to give children credit when it took them several seconds to recognize that one of the operands was a fraction form of one before applying that knowledge. Problems with 5/6 as an operand were used to examine multiplication trials meeting this criterion, because they were presented for all four types of problems. Children met the speed and accuracy criteria on 43% of multiplication trials with one in integer form, 9% with one expressed as a one-digit fraction, 7% with one expressed as a two-digit fraction, and 4% when 5/6 was multiplied by a fraction other than one. Similarly, on division problems, where 1/2 was divided by all four types of operands, children met the speed and accuracy criteria on 26% of trials where one was expressed as an integer, 4% when it was expressed as a one-digit fraction, 4% when it was expressed as a two-digit fraction, and 2% when the divisor did not equal one. This analysis indicated that even with integer forms of one, children used knowledge that it was the identity element on fewer than half of trials. However, they did use that knowledge far more often when one was an integer than when it was a fraction. This again indicated that children possessed knowledge of one that they did not apply on arithmetic problems.

Prediction 3: Estimates of fraction forms of one on number lines will be less accurate than estimates of the integer form of one.

Accuracy of number line estimates was measured by Percent Absolute Error (PAE), calculated as |Actual Magnitude − Participant's Estimate| / Numerical range × 100. For example, if a participant was presented a 0-1 number line and estimated .50 at .49, the PAE would equal |.5 − .49| / 1 = 0.01 × 100 = 1.

Number line estimation PAE was considerably lower (accuracy was higher) when the number one was presented as an integer (PAE = 11%, *SD* = 16%) than when it was presented as a fraction (PAE = 22%, *SD* = 18%), $t(44) = 4.309$, $p < .001$. A one-way repeated measures ANOVA revealed that numerical range also affected PAE, $F(1.263, 55.577) = 26.238$, $p < .001$, $\eta_p^2 = .374$. PAE was lowest on the 0-1 lines (PAE = 11%, *SD* = 12%), intermediate on the 0-2 lines (PAE = 17%, *SD* = 13%), and highest on the 0-100 lines (PAE = 30%, *SD* = 22%).

A one-way repeated measures ANOVA on estimates of the number 50/50, which was presented with all three number lines, revealed a similar effect of the number lines' range, $F(1.777, 78.172) = 19.610$, $p < .001$, $\eta_p^2 = .308$. Estimates on 0-1 lines were the most accurate (PAE = 11%, *SD* = 20%, $p < .001$); those on 0-2 lines were similarly accurate (PAE = 15%, *SD* = 21%, $p = .952$), and both were more accurate than those on 0-100 lines (PAE = 38%, *SD* = 36%, $p < .001$).

A one-way repeated measures ANOVA revealed a similar effect of range of the number line on estimates of the integer one, $F(1.442, 63.451) = 16.617$, $p < .001$, $\eta_p^2 = .274$. Estimates for the integer form of 1 were approximately the same for 0-1 and 0-2 number lines (PAE = 3.6%, *SD* = 15.5% and PAE = 3.7%, *SD* = 9.7% respectively, $p > .05$), but estimates on both lines were more accurate than estimates on the 0-100 number line (PAE = 25%, *SD* = 35%, $p < .001$).

Prediction 4: Students' lack of flexible understanding will extend to answers to explicit questions regarding one as a fraction that require conceptual understanding.

A one-way repeated measures ANOVA on accuracy of answers regarding the four categories of explicit questions about varying forms of one (Table 2) revealed a main effect of category type, $F(3, 132) = 72.378$, $p < .001$, $\eta_p^2 = .622$). Participants were more accurate on Category 1 problems (86% correct) than on Category 2 problems (62% correct, $p = .014$), and they were more accurate on both Category 1 and 2 problems than on Category 3 and 4 problems (33% and 24% respectively, all $p < .05$).

The greater accuracy on Category 1 than Category 2, 3, and 4 problems indicated that students had greater knowledge of fraction forms of one when they involved individual fractions than when they involved fraction arithmetic. The greater accuracy on Category 2 problems than on Category 3 problems indicated higher performance when the standard fraction arithmetic procedure yielded answers literally identical to a response option (or different from them when the correct answer was "false") than when it yielded answers equivalent but not identical to a correct response option. For example, 58% of children answered correctly that 3/4 × 2/2 = 6/8, but only 35% were correct on 5/6 × 2/2 = 5/6, presumably because 10/12 was not identical to 5/6.

Problems involving fraction-by-integer multiplication yielded a similar pattern. For example, 90% of students correctly indicated that 4/9 × 1 = 4/9, an answer that would emerge from multiplying 4/9 either by one in its integer form or as the fraction 1/1. However, only 21% of students correctly indicated that 2/5 × 1 = 4/10, likely because multiplying numerators and denominators by 1 or by 1/1 would not produce the literal answer 4/10. Again, accuracy on all such problems was higher when the standard fraction multiplication procedure yielded an answer identical to the proposed answer than when it yielded an equivalent answer.

Another item examined the frequency with which students treated n and n/n as equivalent when asked for explicit judgments regarding fraction multiplication. Most students (76%) incorrectly judged 4/5 × 2 = 8/10 to be true, replicating and extending observations from Muzheve and Capraro's (2012) classroom study.

Despite these differences in children's accuracy when the standard fraction arithmetic procedure did or did not yield the literal proposed answer, there was no difference in students' confidence in the correctness of their judgments on the two types of problems. The mean confidence ratings were 74% and 75% confidence, respectively, as indicated by placement of the slider on the 0-100 line, $t(47) = -.229$, $p = .82$. A 2 (Proposed answer: correct or incorrect) x 2 (Match between proposed answer and answer yielded by multiplying numerators and multiplying denominators: literal or non-literal) repeated measures ANOVA was conducted on mean confidence in answers among children who generated at least one correct and one incorrect answer). The analysis showed no main effect of either variable. Participants were no more confident in their correct answers ($M = 69.5$) than in their incorrect ones ($M = 70.4$). Confidence also was similar when the standard procedure yielded an answer that literally matched a response option ($M = 68.78$) as when it yielded an equivalent rather than an identical answer ($M = 71.17$).

Discussion

Findings from Study 1 indicated that middle-school students have little flexible understanding of the number one. Their accuracy on all three tasks was far lower with fraction than integer forms of one.

These findings might be interpreted as indicating lack of understanding of fraction equivalence in general rather than lack of knowledge of fraction forms of one. For example, some of the true/false and multiple-choice problems were relevant to assessing understanding of fraction equivalence (e.g., 2/5 × 1 = 4/10), as well as to assessing understanding of the number one. However, other data from Study 1 attest specifically to a lack of understanding of the number one in fraction form. For example, despite the explicit goal of the Common Core Standards that 3rd graders should be able to "locate 4/4 and 1 at the same point of a number line diagram" (CCSSI, 2010, 3.NF.A.3.C, p. 24), 6th-8th graders in the present study were much less accurate in locating fraction than integer forms of one, despite years of fraction instruction after third grade. Moreover, far fewer students (33% versus 60%) correctly answered 8/9 ☐ 8/9 × 11/11 than 5/9 ☐ 5/9 × 1, despite the only meaningful difference between the problems being whether the number one was in fraction or integer form. Understanding numerical equivalence is indeed a weakness in many students' knowledge of rational numbers, but understanding fraction forms of one seems to be a further weakness.

Study 2

Why might children have such weak understanding of fraction forms of one? One plausible contributor is inadequate textbook coverage. Children may encounter few instances of fraction forms of one in their textbooks. To test this possibility, we examined textbook input regarding different forms of one in Study 2.

Textbooks clearly are a major source of mathematical input (Cai, 2014; Valverde et al., 2002). Recent studies have found that more than 90% of fourth and fifth grade teachers use textbooks in more than half of their lessons (Blazar et al., 2019) and that 70% of fraction and decimal arithmetic problems that third through sixth grade teachers assign come from textbooks (Tian et al., 2022). Thus, textbooks form a large part of the math learning environment.

Children's math learning seems to be affected even by seemingly unimportant features of textbooks (for a review, see Siegler et al., 2020). For example, textbook input appeared to contribute to Siegler and Pyke's (2013) surprising finding that despite the standard solution procedure being identical, children's fraction multiplication is considerably more accurate on problems with unequal than equal denominators (i.e., 1/6 × 4/5 versus 1/5 × 4/5). One potential explanation was that few fraction multiplication problems with equal denominators appeared in the three textbook series that were examined. In contrast, many problems in the textbooks involved fraction multiplication with unequal denominators, and many problems involved fraction addition and subtraction with equal denominators (Braithwaite et al., 2017). Children learn these regularities in textbook problems – they predicted far more often that multiplication would be the operation when presented problems with unequal denominators (e.g., 3/5 ☐ 2/3) than when presented problems with equal denominators (e.g., 3/5 ☐ 2/5) (Braithwaite & Siegler, 2018).

Similar findings have emerged in studies of other areas of mathematics, such as mathematical equivalence (e.g., McNeil et al., 2015; Powell, 2012). Analyses of textbooks have shown that the overwhelming majority of whole number arithmetic problems have operations on the left side of the equal sign but not on the right side. This consistent presentation format allows students to conclude that equal signs are an instruction to perform the operation on the left side (e.g., 2 + 3 = ☐) rather than an indicator that the values of the symbols on the two sides are equal (McNeil et al., 2006). However, this conclusion becomes problematic when children encounter operations on both sides of the equal sign (e.g., 2 + 3 = ☐ + 4); in these situations, children often erroneously add only the numbers on the left side of the equal sign or add all of the numbers in the problem (Falkner et al., 1999).

Biases in textbook problems similar to those documented in other areas may contribute to children's difficulty understanding fraction forms of one. To test this hypothesis, in Study 2, we examined three popular textbook series for the frequency of fraction arithmetic problems having an operand equivalent to one and the number of pages with pedagogical content that noted the equivalence of n/n to one. The textbook series -- Houghton Mifflin Harcourt's Go Math! (Dixon et al., 2012), Pearson's enVisionmath (Charles et al., 2012), and McGraw-Hill's Everyday Mathematics (UCSMP, 2015) -- were three of the most widely used series in the US when the database was created in 2016 (see Opfer et al., 2018).

Coding and Analyses

We first examined fraction arithmetic problems in textbooks to determine their frequency of fractions equivalent to one. The database consisted of all fraction arithmetic problems in grade 3-6 textbooks that had two fraction operands and that required children to generate an exact numeric answer. Word problems were excluded from the database because of the difficulty of categorizing many such problems (Geary, 2004). A total of 4,668 problems were included in the database.

The second part of the analysis involved both qualitative and quantitative measures of instruction in the textbooks aimed at fostering flexible understanding of the number one as a fraction. All 1,538 pages that included fraction arithmetic problems were examined.

Instructional content was coded as making direct reference to fractions equivalent to one or not doing so (either not using such fractions or using them to simplify fractions or establish common denominators but not referring to n/n = 1; see the example at the bottom of Table 3). Students who recognized multiplying and dividing by the same number (in this case, multiplying and dividing by "2") as equivalent to multiplying by one could code some problems in that way, but nothing in the exposition called attention to the relation.

Table 3

Coding of Fraction Instructional Content That Did and Did Not Note Fraction Forms of One

Coding	Definition	Example[a]
Did	The instructional content on the page draws attention to the fact that a fraction is equivalent to the integer one, either through explicit statements or through use of an equal sign.	Subtract the part that was eaten from the whole cake. $1 - \frac{7}{10}$ Write 1 as a fraction with denominator 10. $1 = \frac{10}{10}$ $\frac{10}{10} - \frac{7}{10} = \frac{3}{10}$
Did Not	None of the instructional content on the page indicates that n/n is equivalent to one. The instruction may tell students to multiply or divide the numerator and denominator by the same number but does not note that doing so is equivalent to multiplying or dividing by 1.	Simplify: $\frac{8}{10}$ Divide the numerator and denominator by 2. $\frac{8 \div 2}{10 \div 2} = \frac{4}{5}$

[a]Examples were adapted from problems in one of the textbook series.

The second author and a trained research assistant independently coded one textbook series to calibrate the coding scheme; discrepancies were resolved through discussion. Then, the second author and the trained research assistant each coded one of the other textbook series using the agreed upon coding scheme. To check for consistency, each coder independently coded a randomly selected 20% of the pages from the textbook that had been coded by the other coder. Agreement for this coding was 96% (Cohen's Kappa demonstrated that there was moderate agreement between the two raters $\kappa = .673$ (95% CI, .538 to .808), $p < .001$); again, discrepancies were resolved through discussion.

Results

We examined both the percentage of fraction arithmetic problems in the three textbooks that referred to the number one in the form of the fraction n/n and the percentage of pages that included instruction that noted the equivalence of n/n to the integer one.

Fraction Arithmetic Problems Including Fractions Equal to One

Problems that included fraction forms of one (n/n) or that called attention to the equivalence of integer and fraction forms of one were almost non-existent in all three U.S. textbook series: 0.8% (9 problems) in enVisionMATH, 0.0% in Everyday Math, and 0.3% (1 problem) in Go Math!.

Instructional Content Aimed at Developing Flexible Understanding of One

Instructional content also was rarely aimed at fostering flexible understanding of one. Over the four years of instruction, Go Math! included such content on 2% of pages that included fractions content, enVisionmath on 1%, and Everyday Mathematics on 4%.

General Discussion

Flexible understanding of the number one is essential in many areas of mathematics. The present findings, however, suggest that middle-school students' understanding of how to flexibly apply fraction representations of one is very limited. In this concluding section, we discuss children's abilities to represent fraction forms of one, textbook coverage as a potential source of limitations of knowledge about one, and educational implications and limitations of this study.

Students possess factual knowledge that fraction forms of one equal one.

As predicted, most middle-school children correctly answered explicit questions about fraction forms of one as an individual number. For example, 95% of students correctly indicated that "36/36 = 1" is a true statement.

Children fail to apply understanding of fraction forms of one to estimate magnitudes.

The same children did not flexibly apply such knowledge to estimating the magnitude of fraction forms of one. Estimation of fraction forms of one on number lines was considerably less accurate than estimation of integer forms of one on the same lines.

Children fail to apply knowledge of fraction forms of one to fraction arithmetic.

Despite demonstrating knowledge that fraction forms of one equal the integer form when asked directly whether N/N = 1, children solved arithmetic problems far more quickly and accurately when the problems included integer forms of one than when they included fraction forms of it. For example, children were faster and more accurate on problems such as "1 + 1/3" than on "6/6 + 1/3" (M = 55%, 34 seconds and M = 40%, 65 seconds, respectively). Moreover, children were neither faster nor more accurate on problems with fraction operands equivalent to one than on problems where neither operand equaled one.

Instead, children consistently relied on the literal answers yielded by standard fraction arithmetic procedures to judge the accuracy of proposed answers to multiplication and division problems. Equivalence to the proposed answer was insufficient for statements to be judged true. For example, whereas 3/4 × 2/2 = 6/8 was judged true by 58% of students, 5/6 × 2/2 = 5/6 was judged true by only 35%. By contrast, 90% of students correctly indicated that 4/9 × 1 = 4/9. Judgments of correctness of answers to fraction arithmetic problems thus reflected reliance on literal identity of answers rather than flexible understanding of the number one.

Textbooks may contribute to weak understanding of fraction forms of one.

Our analysis of the third to sixth grade volumes of three popular contemporary US math textbook series revealed strikingly little emphasis on fraction forms of one. Across the twelve textbook volumes (four grades × three textbook series), only ten problems (< 1% of all fraction arithmetic problems) explicitly used fraction forms of one (n/n) or referred to the equivalence of fraction and integer forms of one. Prior work has shown that distributions of practice problems are related to children's performance on similar problems (McNeil et al., 2015; Tian et al., 2021). With virtually no practice nor pedagogical content in textbooks directly relevant to fraction operands equivalent to one, children acquire only weak understanding of the varied uses of one in fraction magnitude representation and arithmetic.

The reasons for this paucity of coverage of fraction forms of one are unclear. Perhaps, textbook writers reasoned that providing fraction arithmetic problems with one as a fraction operand (e.g., 6/6 + 1/3) would be too easy, because students would recognize a fraction equivalent to one and apply a shortcut. This would be consistent with Braithwaite and Siegler's (2018) finding that textbooks rarely present such problems with one in integer form (e.g., 1 + 1/3). The present findings suggest, however, that if fear of the problems being too easy was the reason for the scarcity of such problems, the concern was groundless. Students were no more accurate or faster on problems with a fraction operand equal to one as on problems with two fraction operands unequal to one. Students did apply a shortcut when problems involved one as an integer (e.g., 1 + 1/3), answering such problems faster and more accurately than other fraction addition problems, but this was not the case with fraction forms of one. The use of shortcuts on problems such as 1 + 1/3 shows that students can employ flexible thinking on rational number tasks, but usually fail to do so with fraction forms of one.

Our analyses of textbooks revealed many missed opportunities to improve understanding of fraction forms of one. Textbook presentations repeatedly reminded students to multiply the numerator and denominator by the same number to generate an equivalent fraction, but they rarely explained why multiplying the numerator and denominator by the same number would have this effect. The focus on multiplying by independent whole numbers may have had the unintended consequence of instilling a misconception that multiplying a number by either "n" or "n/n" would have the same effect.

The identity produced by multiplying or dividing the numerator and the denominator of a fraction by the same number seems to be far from obvious to most learners. This is understandable; after all, adding the same number to the numerator and denominator of a fraction changes the value of the original fraction. Without instruction including good explanations and multiple practice problems, many, perhaps most, students do not flexibly apply their knowledge of integer forms of one to fraction forms. The present results suggest that these students are not receiving the instruction they need to make the connection.

Educational Implications

What if textbooks included a greater amount of explicit instruction and practice on arithmetic problems with fractions equivalent to one (e.g., 6/6 + 1/3)? We believe that this small curricular change would improve the flexibility of children's understanding of one, especially if such problems were accompanied by instruction noting the equivalence of integer and fraction forms of one and clear, persuasive illustrations and explanations of the equivalence and its implications for magnitude estimates and arithmetic with fraction forms of one. Other small curricular changes impact mathematics understanding. McNeil and colleagues (2015) found that second graders' understanding of mathematical equality improved considerably by presenting problems rarely seen in their textbooks with operations on the right side of the equal sign (e.g., ☐ = 3+2) and by replacing the equal sign with the words "the same as."

Perhaps, if textbooks had more problems with one as a fraction operand and other problems that could be solved by shortcuts as well as standard procedures, students might begin to think more flexibly about the quantities involved in the problems. Future research should examine the effects of providing more direct instruction about the role of both integer and fraction forms of one as identity elements for multiplication and division, as well as explanations of *why* multiplying and dividing by fraction forms of one does not change the value of the number being multiplied and divided. Such small changes could have large effects on children's flexible understanding of the number one.

Limitations of the Present Study

The present findings do not demonstrate a causal connection between input from textbooks and children's learning. Future research should include randomized controlled trials to investigate the effects of providing students more practice with fraction operands equal to one, more pedagogical content drawing students' attention to effects of multiplying and dividing with varied forms of one, and clear, persuasive explanations of why fraction forms of one have the same effect on multiplication and division as integer forms of one.

The current study limited its investigation of flexible understanding of the number one to fraction arithmetic and magnitude representation. However, effects of such flexible understanding of one might also be important for other areas of mathematics. For example, flexible understanding of the number one likely helps children simplify algebraic expressions (Yakes & Star, 2011). Similar benefits might be seen in other areas (e.g., calculus, statistics, etc.) in which flexible understanding of one is foundational.

Another possible limitation is that this research was conducted during the global COVID-19 pandemic, which necessitated remote data collection. Future studies should attempt to replicate the phenomena observed here under typical classroom and laboratory circumstances.

Conclusions

The current findings demonstrated that many US middle school students do not apply their knowledge of integer forms of one to thinking flexibly about fraction forms of one. These findings point to a larger issue. If children are blindly executing arithmetic procedures without concern for the numbers involved, they are not reasoning quantitatively. Without such quantitative reasoning, mathematics learning often degenerates into a tedious process of memorizing rules and procedures as arbitrary facts. One step toward building mathematical flexibility might be to help children acquire understanding of the number one in its many forms and ease in translating among those forms. This might help them understand that any number, not just one, can be represented in infinite ways, a concept at the core of rational number understanding.

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Supplementary Materials

The Supplementary Materials contain math problems used for experimental stimuli, including fraction arithmetic and multiple-choice problems organized by the relevant categories (for access see Index of Supplementary Materials below).

Index of Supplementary Materials

Schiller, L. K., Fan, A., & Siegler, R. S. (2022). *Supplementary materials to "The power of one: The importance of flexible understanding of an identity element"* [Experimental stimuli]. PsychOpen GOLD. https://doi.org/10.23668/psycharchives.8194

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